Extensions and Variations of the Two-Person Game on Graphs

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Introduction

A *graph* $G = (V, E)$ is an ordered pair of disjoint sets V and E such that E is a subset of the set $\binom{V}{2}$ of unordered pairs of V. V denotes the set of *vertices* and E denotes the set of *edges* of G. A key concept of graph theory is the theory of graph coloring which has its roots in the famous *four color conjecture* raised by Francis Guthrie 1852. In order to color the regions of a map, he proposed that at most four colors are required so that neighboring regions are assigned different colors. A *vertex-coloring* of a graph $G = (V, E)$ is a mapping $c : V \to S$ such that $c(a) \neq c(b)$ whenever $(a, b) \in E$. The elements of the set $S = \{1, ..., k\}$ are called the available *colors*. The *chromatic number* $\chi(G)$ of a graph G is the smallest integer k such that G is colorable.

A modern field of graph coloring is the *two-person game on graphs*, which goes back to the work of Bodlaender in [3], 1991. Bodlaender defined the two-person game on graphs as follows: Let $G = (V, E)$ be a graph and C a given set of colors. The two players Alice and Bob take turns assigning the vertices of G colors from C such that adjacent vertices are assigned different colors. Alice starts the game. She wins, if all vertices of G are colored. Otherwise, if there exists at least one vertex that cannot be colored with a color from C, Bob wins. The *game chromatic number* of G, denoted by $\gamma(G)$, is the least cardinality of C, such that there exists a winning strategy for Alice. Obviously, it holds

$$\chi(G) \leq \gamma(G) \leq |V|.$$

The aim of this thesis is to generalize Bodlaender's idea and to establish three extensions of the two-person game on graphs.

1. *The circular two-person game on graphs* (Chapter 2)

2. *The two-person game on weighted graphs* (Chapter 3)

3. *The general asymmetric game on graphs* (Chapter 4)

The first extension is based on the idea of the circular coloring of graphs, introduced by Zhu in [22] (see definition 1.2.1) as an equivalent definition to Vince's *star chromatic number*, [20]. By bringing together the two areas *the two person game* and *the circular coloring*, we work out the *circular two-person game* in chapter 2 *(The Circular Two-Person Game on Graphs)* . While in Bodlaender's two-person game the given colors are natural numbers, in the circular two-person game Alice and Bob assign to the vertices of a graph unit length arcs on a circle C^r with circumference $r \in \mathbb{R}^+$. Note that the arcs of adjacent vertices are not allowed to intersect in a feasible circular coloring. We define the *circular game chromatic number* $\gamma_c(G)$ of a graph G as the infimum of those r for which there exists a winning strategy for Alice on C^r. While $\gamma(G)$ is a natural number, for the new parameter it holds $\gamma_c(G) \in \mathbb{R}^+$. We in particular show that

$$\gamma_c(G) \leq 2\Delta(G),$$

where $\Delta(G)$ is the maximum degree of G. The main difficulty in working out a winning strategy for Alice is that one must figure out Bob's optimal strategy in terms of the arcs he destroys on C^r, that is, they cannot be assigned to any vertices. Thus, the new game sheds some new light on the respective strategies of the players.

A continuative aim of chapter 2 is to discuss the new parameter for some relevant classes of graphs. We work out winning strategies for Alice for the class of complete graphs and cycles in sections 2.1 and 2.2, respectively. Moreover, in section 2.3 we prove for a complete multipartite graph $K_{s_1,\dots,s_n}$ on n independent sets $S_1, \dots, S_n$ with $|S_i| = s_i$, that $\gamma_c(K_{s_1,\dots,s_n}) \leq 3n - 2$. In 2.3.1 we look more closely at the circular game chromatic number of a complete bipartite graph K_{s_1,s_2} minus a perfect matching M and establish that

$$\gamma_c(K_{s_1,s_2} - M) \geq \gamma_c(K_{s_1,s_2}).$$

Afterwards, we turn our attention to the well known *activation strategy*, introduced by Kierstead in [11]. A lot of strong results have been won using

Kierstead's activation strategy, which is a winning strategy for Alice for *the marking game*, a simplified version of Bodlaender's two-person game (see page 8). In [26] Zhu pointed out the notion of the marking game and proved that the marking game number of a graph G, denoted by $col_g(G)$, provides an upper bound of the game chromatic number such that it holds

$$\gamma(G) \leq col_g(G).$$

We prove that the same conclusion can be drawn for the circular version of the game such that for every graph G it holds

$$\gamma_c(G) \leq col_g^c(G),$$

where we define $col_g^c(G)$ as the *circular marking game number* of G. In particular, we generalize the activation strategy for the case of circular coloring and show that the circular game chromatic number is bounded by a parameter which we define as the *circular rank* of a graph G. In particular, we use this result and prove for the upper bound for planar graphs that

$$col_g^c(G) \leq 34.$$

Finally, we turn our attention to the class of cactuses where we use techniques by Sidorowicz in [19]. We prove that $col_g^c(G)$ is bounded by 8.

Independently Zhu and Lin also dealt with the circular game chromatic number in [27].

Our second extension of the two-person game is the subject of chapter 3 *(The Two-Person Game on Weighted Graphs)*. We consider a *vertex-weighted graph* which is a triple $(G, w) = (V, E, w)$ where w is a mapping $w : V \to \mathbb{N}$ that assigns to every vertex $x \in V$ a positive integer $w(x) \geq 1$. A coloring of vertex-weighted graphs requires the additional condition $c(x) \geq w(x)$, where $c : V \to \mathbb{N}$ is the coloring mapping. Thus, we define *the game chromatic number of weighted graphs*, denoted by $\gamma(G, w)$, as the smallest amount of given colors, such that there exists a winning strategy for Alice for the coloring game on a weighted graph (G, w). Obviously, if the mapping w takes the constant value 1, it holds that $\gamma(G) = \gamma(G, w)$.

The consideration of the two-person game on weighted graphs provides a generalization of Bodlaender's two-person game because for a weighted graph $(G, w) = (V, E, w)$ it is assumed that $w(v) \geq 1$ for all $v \in V$. The difficulty in carrying out the two-person game on weighted graphs is that during the whole game Alice and Bob have to take not only the structure of the graph but also the vertex-weights into account. In particular, they have to make sure that the color $c(v)$ of a vertex $v \in V$ is equal or greater than its weight $w(v)$, while figuring out their optimal strategies.

It is well known and obvious that for the chromatic number of a graph G it holds that $\chi(G) \leq \chi(G, w)$. One may conjecture that the same conclusion can be drawn for the two-person game, such that $\gamma(G) \leq \gamma(G, w)$ because $w(x) \geq 1$ for all $x \in V$. However, in section 3.4 we work out a surprising result and construct graphs with the property $\gamma(G, w) < \gamma(G)$. Furthermore, we characterize this class of graphs in proposition 3.4.3.

Moreover, we wish to investigate the new parameter for some relevant classes of graphs considering certain distributions of vertex-weighs.

In section 3.1 the game chromatic number of a weighted complete graph (K_n, w) on n vertices is investigated. After determining $\gamma(K_n, w)$ for all values of w, we establish the relation between the required number of colors for a feasible coloring and $w_{\max}(K_n, w)$ which is the maximum vertex-weight of the graph. Furthermore, we analyze the new parameter for a complete graph under the assumption $w : V \to \{k, l\}$ where $k, l \in \mathbb{N}$ and $k \neq l$.

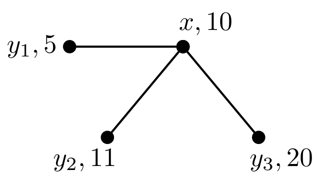
In the next section we turn our attention to the class of complete multipartite graphs relating to certain distributions of vertex-weights. We give a lower as well as an upper bound for the entire class. In addition we restrict the weight function such that $w(S_i) \in \{k, l\}$ for $k, l \in \mathbb{N}$ and $k \neq l$, while assuming that vertices belonging to the same independent set achieve the same weight. In particular, Alice's strategy has to be adapted for the cases $s_i \geq 3$ and $s_i \geq 4$ for all $i \in \{1, ..., n\}$. Afterwards, this idea is being generalized such that we work under the assumption that $w(S_i) \neq w(S_j)$ for all $i, j \in \{1, ..., n\}$ and vertices belonging to the same independent set achieve the same weight. Finally, we drop the assumption that vertices contained in the same independent set have the same weight and analyze the game for the case that $w : V \to \{k, l\}$

where $w(S_i) \in \{k, l\}$ for $k \neq l$.

Afterwards, in section 3.3 we proceed with the game chromatic number of weighted cycles. First, we give a lower bound as well as an upper bound for all values of w and give conditions when they are attained. An interesting result is obtained, if we consider $(k_1, ..., k_m)$-*alternating-weighted* cycles, where for each two vertices x_i and x_j with distance m it holds $w(x_i) = w(x_j)$.

In section 3.5 we analyze the game on weighted trees applying an algorithm of Faigle, Kern, Kierstead and Trotter in [14]. They estimated the maximum number of colored adjacent vertices of an uncolored vertex during the game by 3. In particular, for determining the upper bound of $\gamma(T, w)$, where (T, w) is a weighted tree, we need to observe the weights of the *leaves*, as well as the *interior-vertices* of a tree.

Section 3.6 deals with a generalization of Kierstead's activation strategy by admitting vertex-weights. In particular, we show that the game chromatic number of a weighted graph is bounded by a parameter which we call *weighted rank* and give an upper bound of the game chromatic number for weighted planar graphs. The main difficulty in contrast to the ordinary two-person game is that for a vertex $x \in V$ a neighbor $y \in N(x)$ with $w(y) > w(x)$ is not necessarily a threat for x. The following example demonstrates this. Let $\{y_1, y_2, y_3\} \in N(x)$ with $w(x) = 10$, $w(y_1) = 5, w(y_2) = 11$ and $w(y_3) = 20$. Vertex y_2 with weight 11 seems to be harmless for x, because $w(x) = 10$ and y_2 has to be colored with a color greater or equal 11. However, if Bob colors y_1 with the color 10 and y_2 with the color 11, then x is being *attacked* twice. On the contrary, x cannot be attacked by y_3 since $|N(x)| = 3$.



The subject of chapter 4 *(General Asymmetric Games on Graphs)* is a variation of Bodlaender's two-person game going back to the work of Kierstead. In [17] he introduced the *asymmetric game on graphs*. The basic difference to the ordinary game is that the players color several vertices in a row instead of one vertex each time they take turns; in particular, Alice colors a and Bob b vertices for $a, b \in \mathbb{N}$ and $a, b \geq 1$ (see page 9). Further, a turn is not being completed as soon as either every vertex is colored or a feasible coloring of the graph is not possible. For a graph $G = (V, E)$ the *asymmetric game chromatic number*, denoted by $\gamma(G; a, b)$, is defined as the least integer s such that there is a winning strategy for Alice in the asymmetric game using s colors. Note that for $a = b = 1$ it holds $\gamma(G) = \gamma(G; a, b)$.

We go a step further by considering the case that the number of the moves varies each time the players take turns. The crucial point is to define the sets of moves as m-tuples $\bar{a} := (a_1, ..., a_m)$ and $\bar{b} := (b_1, ..., b_m)$, where x_i vertices are colored during the ith turn for $x \in \{a, b\}$ and $i \in \{1, ..., m\}$. We call this new game *general asymmetric game* and define the *general asymmetric game chromatic number* of a graph $G = (V, E)$, denoted by $\gamma_g(G; \bar{a}, \bar{b})$, as the least integer s such that there exists a winning strategy for Alice when the general asymmetric game is played on G using s colors. Obviously, for $x_i = x_j$ for all $i, j \in \{1, ..., m\}$ and $x \in \{a, b\}$ it holds

$$\gamma(G; a, b) = \gamma_g(G; \bar{a}, \bar{b}).$$

This new consideration provides a more general characterization of the ordinary asymmetric game since the number of the moves is variable. It is of our interest to investigate $\gamma_g(G; \bar{a}, \bar{b})$ for certain distributions of the m-tuples for the class of cycles, complete multipartite graphs and forests in sections 4.1, 4.2, 4.3, respectively.

Considering a cycle C_n on n vertices, we work out a winning strategy for Alice for all values a_i and b_i; in particular, we show that for $a_1 \geq \lceil \frac{n}{3} \rceil$ Alice fixes the coloring of C_n for n even, after her first turn, if 2 colors are given. For the case $a_1 < \lceil \frac{n}{3} \rceil$ we show that there does not exist an optimal strategy for Alice, i.e., Bob achieves his best case.

Further, we turn our attention to the general asymmetric game chromatic number of complete multipartite graphs. We work out the optimal strategy for Alice and conclude Bob's worst case for all values a_i and b_i. Precisely, for a complete multipartite graph $K_{s_1,...,s_n}$ on n independent sets, we prove a winning strategy for Alice for $a_1 \geq n$, where the coloring of the graph is fixed after her first turn. A more interesting result is obtained, if we assume that $a_1 < n$. We prove an optimal strategy for Alice and determine the required number of colors for the cases that Alice or Bob fixes the coloring of the graph, respectively.

For the purpose of investigating the new parameter for the class of forests, we apply techniques by Kierstead, which he worked out for the ordinary asymmetric game in [17]. We give an upper bound for the case $a_{i+1} \geq b_i$ for all $i \in \{1, ..., m\}$ as well as for the case that there exists one and only one $j \in \{1, ..., m\}$ with $a_{j+1} < b_j$. Furthermore, we determine lower bounds while working under the assumption that $a_1 < 2b_1$ and $a_1 = ... = a_m$ where b_i is strictly increasing for $i \in \{1, ..., m\}$. In particular, we prove a winning strategy for Bob and calculate after how many turns he wins the game for some relevant distributions of $\bar{b} = (b_1, ..., b_m)$.

Chapter 1

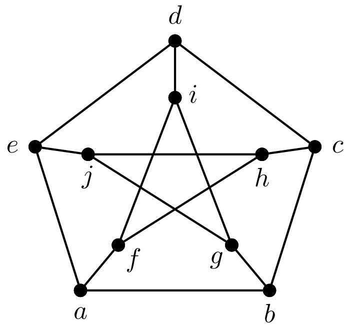
Preliminaries

In this chapter we will present some basic definitions and set up notations and terminologies required throughout the thesis. In section 1.2 (Graph Coloring) we will introduce the notions of graph coloring and circular graph coloring, while summarizing without proofs some relevant results. In particular, we will look more closely at vertex-coloring of weighted graphs. Section 1.3 (Games on Graphs) will be devoted to games on graphs. After a brief exposition of the two-person game, we will give the definition of a simplified version of it, called the marking game. Finally, the asymmetric game will be defined. Specific terms and definitions will be introduced in the respective chapters.

1.1 Basic Definitions

Definition 1.1.1. A *graph* is a pair $G = (V, E)$ of sets such that $E \subseteq \binom{V}{2}$. V denotes the set of *vertices* and E denotes the set of *edges*. We will write $V(G)$ and $E(G)$ instead of V and E if it is not clear from the context. *graph*

The figure below shows the well known *Petersen graph* with vertex set $V = \{a, b, c, d, e, f, g, h, i, j\}$ and edge set $E = \{(a, b), (a, e), (a, f), (b, c), (b, g), (c, h), (c, d), (d, i), (d, e), (e, j), (f, h), (f, i), (g, i), (g, j), (h, j)\}$.



adjacent *neighbor* *incident* *independent* *matching*

Definition 1.1.2. Let $G = (V, E)$ be a graph. Two vertices $a, b \in V$ are *adjacent* if (a, b) is an edge of G, that is, $(a, b) \in E$. Vertex b is also called a *neighbor* of a and vice versa. We denote the set of all neighbors of b by $N(b)$. A vertex $a \in V$ is called *incident* with an edge $e \in E$ if $a \in e$. Two edges $e_1 \neq e_2$ are called *adjacent* if $e_1 \cap e_2 \neq \emptyset$. We call pairwise non-adjacent vertices or edges *independent*. Let $G = (V, E)$ be a graph. A set M of independent edges is called a *matching*. A *perfect matching* is a matching M such that every vertex of G is adjacent to an edge of M.

Throughout the thesis we consider undirected graphs, which means that for every edge (a, b) it holds that $(a, b) = (b, a)$. Edges of the form (x, x) and *multiple edges*, where there exist several edges between the same two vertices, are not allowed. Furthermore, we consider non-empty finite graphs, which means that for the set V it holds that $V \neq \emptyset$ and V is finite.

degree *k-regular*

Definition 1.1.3. Let $G = (V, E)$ be a graph and $a \in V$. The *degree* of a is the number of the neighbors of a and is denoted by $d(a)$. The maximum degree of G is defined as $\Delta(G) := \max\{d(a) \mid a \in V\}$. A graph is called k*-regular* if all vertices of G have degree k.

The Petersen graph is 3-regular since each vertex has degree 3.

induced subgraph

Definition 1.1.4. A graph $G' = (V', E')$ is called a *subgraph* of $G = (V, E)$, and G a *supergraph* of G', if $V' \subseteq V$ and $E' \subseteq E$. In this case we write $G' \subseteq G$. If $G' \subseteq G$ and G' contains all edges $(a, b) \in E$ with $a, b \in V'$, then G' is called *induced subgraph* of G.

Definition 1.1.5. A graph $G = (V, E)$ is called *connected* if for every partition of its vertex set into two non-empty sets X and Y there is an edge with one end in X and one end in Y; otherwise, the graph is disconnected. Let $G' \subset G$ be a maximal connected subgraph of G. Then G' is called a *component* of G.

connected

component

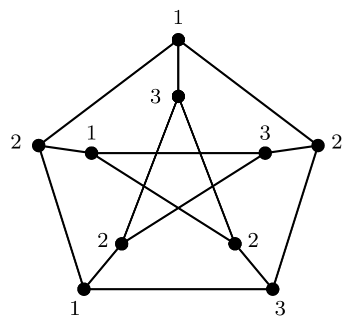
1.2 Graph Coloring

Let $S = \{1, ..., k\}$ for $k \in \mathbb{N}$. A *vertex-coloring* of a graph $G = (V, E)$ is a mapping $c : V \to S$ such that $c(u) \neq c(v)$ whenever $(u, v) \in E$. The elements of S are the available colors. The *chromatic number* $\chi(G)$ is the smallest integer k such there is an S with $|S| = k$ and G can be colored with colors from S.

vertex-coloring

$\chi(G)$

Example: Below we give a coloring for the Petersen graph, where the numbers indicate the respective colors of the vertices.



Circular Coloring of Graphs

Besides the vertex-coloring of graphs there exists a more general coloring, named *r-circular coloring*, which provides a natural generalization of the vertex-coloring. The basic difference is that in case of circular coloring we assign open unit length arcs on a circle C^r with circumference $r \in \mathbb{R}^+$ instead of colors. The object of the circular coloring is to figure out the least circumference r of C^r such that a feasible coloring of the vertices of the graph G is possible. Hence, the result is not necessary a positive integer. In [22] Zhu defined the *r-circular coloring* as follows:

r-circular coloring

$\chi_c(G)$

Definition 1.2.1. Let $r \geq 2$ be a real number, and let C^r denote a circle of length r. An *r-circular coloring* of a graph $G = (V, E)$ is a mapping f which assigns to each vertex x of G an open unit length arc $f(x)$ of C^r such that $f(x) \cap f(y) = \emptyset$ whenever (x, y) of G. The *circular-chromatic number* $\chi_c(G)$ of G is equal to the infimum of those r for which G has an r-circular coloring.

For a graph G the following well know relation holds:

$$\chi(G) - 1 < \chi_c(G) \leq \chi(G).$$

length of an arc

Throughout the thesis we will denote an arc on C^r by $b_{(x_1,x_2)}$ and characterize it by its initial-point x_1 and end-point x_2 on C^r in the clockwise direction, where $x_1, x_2 \in \mathbb{R}^+$. We define the *length* of an arc $b_{(x_1,x_2)}$ by

$$l(b_{(x_1,x_2)}) := \begin{cases} x_2 - x_1 \text{ if } x_2 > x_1, \\ r - (x_1 - x_2) \text{ else.} \end{cases}$$

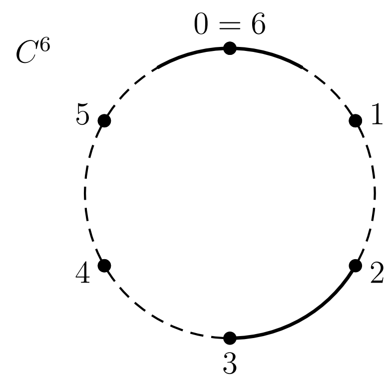
distance between two arcs

In particular, for $b_{(x_1,x_2)} \cap b_{(y_1,y_2)} = \emptyset$ and $x_1 < y_1$ let

$$dist(b_{(x_1,x_2)}, b_{(y_1,y_2)}) := \begin{cases} \min\left\{(y_1 - x_2), (r - (y_2 - x_1))\right\} \text{ for } 0 \notin b_{(y_1,y_2)} \\ \min\left\{(x_1 - y_2), (y_1 - x_2)\right\} \text{ else} \end{cases}$$

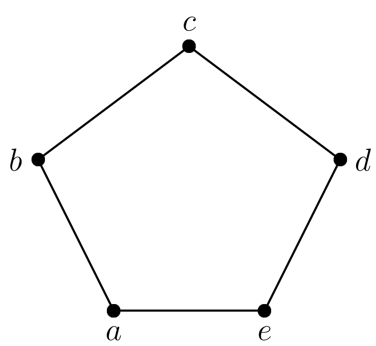
be the distance between the arcs $b_{(x_1,x_2)}$ and $b_{(y_1,y_2)}$.

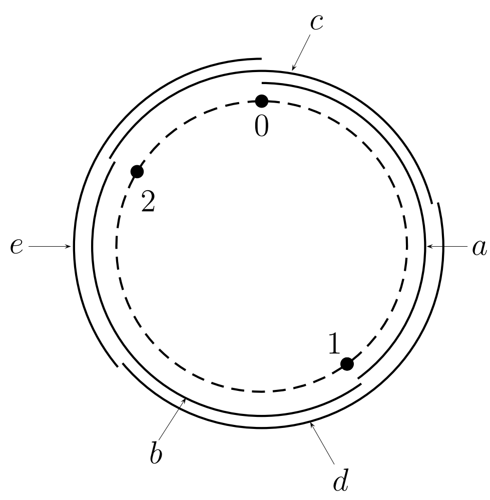
Example: Consider the circle C^6 below and the arcs $b_{(2,3)}$ and $b_{(5.5,0.5)}$. Then $dist(b_{(2,3)}, b_{(5.5,0.5)}) = \min\{(2 - 0.5), (5.5 - 3)\} = 1.5$.



The notion of the circular coloring is based on the idea of the (k, d)-*coloring* and the corresponding *star chromatic number* $\chi^*(G)$, which was introduced by Vince in [20]. Bondy and Hell showed in [6] that the infimum of $\chi^*(G)$ is attained, such that it can be replaced by the minimum. Moreover, they proved that $\chi^*(G)$ is a rational number. In [22], Zhu pointed out two alternate definitions, namely the r-circular coloring (see definition 1.2.1) and the r-*interval coloring*, and showed the equivalence of all three definitions. We will give a detailed insight of the other two definitions in the appendix (A.1). For our purposes, we will consider the notion of the r-circular coloring of graphs. For simplicity of notation we write circular coloring instead of r-circular coloring if it is clear from the context.

Example: Consider the graph below. While this graph has chromatic number 3, its circular chromatic number is $2,5$. Below we give a $2,5$-circular coloring of this graph.





Coloring of Weighted Graphs

Definition 1.2.2. A graph $(G, w) = (V, E, w)$ is called *weighted* if there exists a mapping $w : V \to \mathbb{N}$ which assigns to each vertex x a *vertex-weight* $w(x) > 0$. *weighted graph*

Note that if the mapping w takes the constant value 1, then we write G instead of (G, w).

vertex-coloring of weighted graphs
$\chi(G, w)$

Definition 1.2.3. Let $(G, w) = (V, E, w)$ be a weighted graph and let $S = \{1, ..., k\}$ for $k \in \mathbb{N}$. A *vertex-coloring* of a weighted graph $(G, w) = (V, E, w)$ is a mapping $c : V \to S$ such that $c(x) \geq w(x)$ for all $x \in V$ and $c(u) \neq c(v)$ whenever $(u, v) \in E$. The *chromatic number of a weighted graph* $\chi(G, w)$ is equal to the minimum integer k such that there exists a feasible coloring $c : V \to \{1, ..., k\}$ of (G, w).

Example: Consider the weighted graph (G, w) below. Since there exist two adjacent vertices with vertex-weight 5 at least 6 colors are required for achieving a feasible coloring of G. If we assume that each vertex has weight 1, then 3 colors suffice for coloring G.

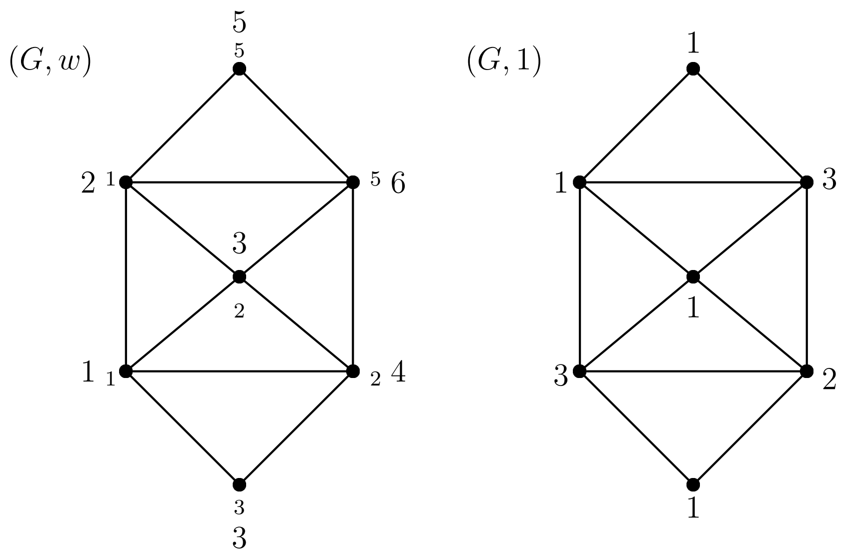


FIGURE: Left: a weighted-coloring where the small-size numbers indicate the vertex-weights while the big-size numbers stand for the colors. Right: a coloring with $w : V \to \{1\}$.

The theory of weighted graphs has a lot of applications, in particular in daily life. For example one can model a road traffic as a weighted graph by assigning the circulation of the traffic from road i to road j to the vertex $v_{i,j}$, where

- the vertex-weight $w(v_{i,j})$ describes the required duration of a green phase and

- two vertices $v_{i,j}$ and $u_{k,l}$ are connected with each other if an overlapping of the corresponding green phase is not allowed and causes an accident.

The chromatic number of the corresponding graph represents the shortest period for a complete traffic control system. See the following road intersection and the corresponding graph.

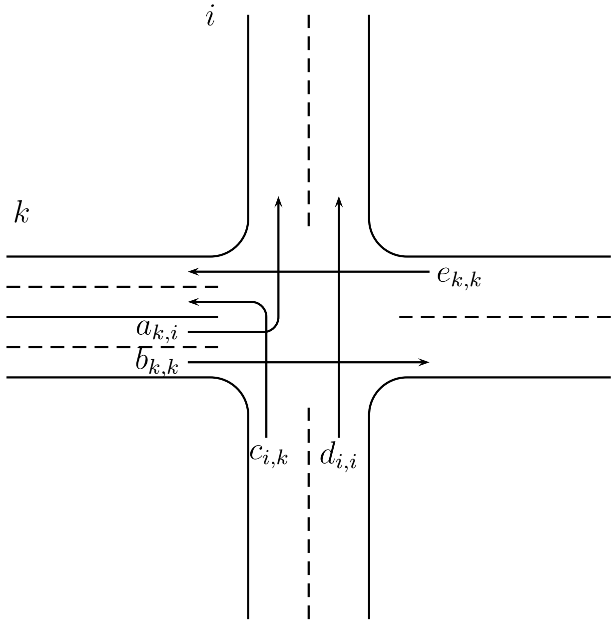


FIGURE (a): A road intersection with $w(b_{k,k}) = w(e_{k,k}) = 4, w(a_{k,i}) = 2$ and $w(c_{i,k}) = w(d_{i,i}) = 7$.

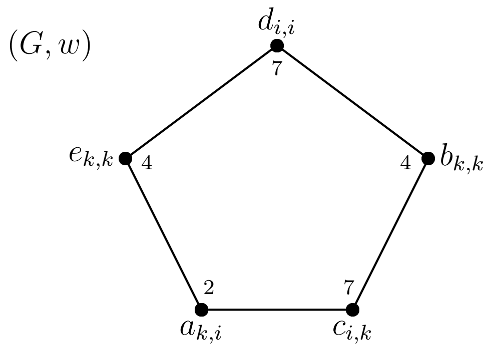


FIGURE (b): The corresponding graph for the road intersection above, where the numbers stand for the vertex-weights.

1.3 Games on Graphs

The Two-Person Game on Graphs

In [3] Bodlaender introduced the *two-person game on graphs* as follows:

the two-person game

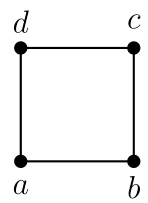
$\gamma(G)$

Let $G = (V, E)$ be a graph and C be a given set of colors. Two players Alice and Bob take turns alternately assigning the vertices of G colors from C, such that adjacent vertices get distinct colors. Alice starts and wins the game if all vertices of G are colored, otherwise, Bob wins. The *game chromatic number* of G, denoted by $\gamma(G)$, is the least cardinality of C, such that there exists a winning strategy for Alice.

Obviously, since Alice and Bob are competitive it holds

$$\chi(G) \leq \gamma(G) \leq \Delta(G) + 1.$$

Example: Consider the two-person game on the graph G below, where obviously $\chi(G) = 2$.



If the two-person game is played on G, 2 colors do not suffice in order to achieve a feasible coloring of G. Assume that 2 colors are given. By the structure of the graph Alice is indifferent which vertex to color first. Thus, without loss of generality assume that she starts the game by coloring vertex a with the color 1. Suppose Bob colors vertex c with the color 2. Since the vertices b and d are adjacent to a and c they cannot be colored neither with 1 nor 2. Thus, Bob wins the game. However, it holds $\gamma(G) = 3$.

The Marking Game on Graphs

In [14] Faigle, Kern, Kierstead and Trotter informally dealt with a simplified version of Bodlaender's two-person game, the *marking game on graphs*. In [26] Zhu formally introduced it as follows:

Let $G = (V, E)$ be a graph and let $t \in \mathbb{N}$ be a given integer. Alice and Bob take turns marking vertices from the shrinking set of unmarked vertices with Alice playing first. This results in a linear ordering L of the vertices with $x < y$ if x is marked before y. The orientation $G_L = (V_L, E_L)$ of $G = (V, E)$ with respect to L is defined by $E_L = \{(v, u) \mid (v, u) \in E \text{ and } v > u \text{ in } L\}$. The *score* of the game is defined by $1+\Delta^+_{G_L}$, where $\Delta^+_{G_L}$ is the maximum number of marked neighbors of an unmarked vertex during the game. Alice wins the game if the score is at most the given integer t. The *marking game number* $col_g(G)$ of G is the least integer t such that Alice has a winning strategy for the marking game played on G.

the marking game

the score

$col_g(G)$

Suppose $col_g(G)$ colors are given. If Alice follows her optimal strategy of the marking game and colors the vertex by First-Fit which is to be chosen, she wins also the coloring game. Thus, this parameter provides an upper bound for the game chromatic number of a graph such that it holds

$$\gamma(G) \leq col_g(G).$$

The Asymmetric Game on Graphs

In [17] Kierstead introduced a modified version of Bodlaender's two-person game on a graph G that differs from it in the following way:

Let $G = (V, E)$ be a graph. Further let C be a given set of colors and $a, b \in \mathbb{N}$. Each time Alice and Bob take turns Alice colors a and Bob b vertices in a row. Alice starts coloring and wins the game if there is a feasible coloring of G with the given set C. Otherwise, Bob wins. Note that if all vertices have been colored, the respective player does not have to complete the turn. Obviously, for $a = b = 1$ we have the regular two-person game on graphs.

The *asymmetric game chromatic number*, denoted by $\gamma(G; a, b)$, is the least cardinality of C such that there is a winning strategy for Alice in the asymmetric coloring game with the given set C of colors.

$\gamma(G; a, b)$

Consider the graph G in the example on page 8. We can conclude that for $a > 1$ the coloring of the graph is fixed after Alice's first turn, if she colors adjacent vertices with 1 and 2 or if she colors alternate vertices with the same color. Thus, it holds $\gamma(G; a, b) = 2$.

Chapter 2

The Circular Two-Person Game on Graphs

We gave the definition of the two-person game and the game chromatic number $\gamma(G)$, which was introduced by Bodlaender in [3], on page 8. In this chapter we intend to generalize Bodlaender's idea by working out a combination of the two-person game and the circular coloring of graphs (see page 4). We will introduce the *circular two-person game on graphs* and define the *circular game chromatic number*. Afterwards, the new parameter will be investigated for the class of complete graphs, complete multipartite graphs, complete bipartite graphs minus a perfect matching, cycles, cactuses and planar graphs. As mentioned in the introduction, Zhu and Lin independently also worked on the circular two-person game in [27].

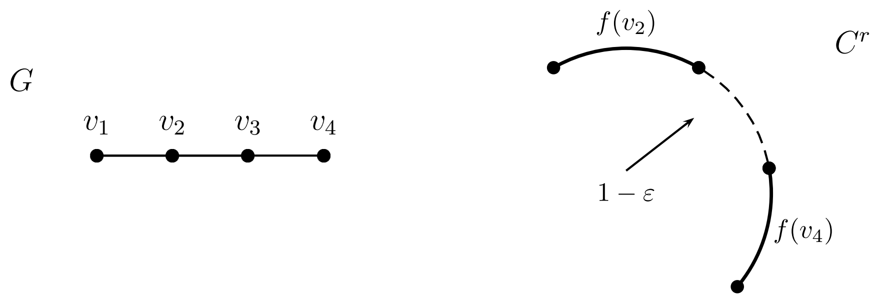
The Circular Game Chromatic Number

Consider the following two-person game: Let $G = (V, E)$ be a graph and let C^r be a circle with circumference $r \in \mathbb{R}^+$. Two players Alice and Bob take turns coloring vertices of G from the shrinking set U of uncolored vertices with Alice coloring first. In each move the respective player assigns to any vertex $v \in U$ an open unit length arc $f(v)$ such that $f(u)$ and $f(v)$ must not overlap whenever $(u, v) \in E$. Alice wins if every vertex can be colored with an arc on C^r. Otherwise, Bob wins.

The *circular game chromatic number* $\gamma_c(G)$ of a graph $G = (V, E)$ is equal to the infimum of those r for which there exists a winning strategy for Alice on C^r.

This new game turns out to be a natural generalization of Bodlaender's two-person game with $\gamma_c(G) \in \mathbb{R}^+$ and $\gamma(G) \in \mathbb{N}$. The basic difference is that the vertices of G are being assigned arcs instead of positive integers. Our example below demonstrates that not only the choice of the vertices to color is decisive but also the choice of the arcs to assign on the given C^r.

Example: Consider the coloring of the graph G on vertices $\{v_1, v_2, v_3, v_4\}$ where $(v_1, v_2), (v_2, v_3), (v_3, v_4) \in E$. By symmetry there are two opportunities for Alice's first turn; either she colors a vertex from $\{v_1, v_4\}$ or $\{v_2, v_3\}$. However, after Alice's first turn at least two consecutive vertices remain uncolored. Obviously, the worst case occurs, if after Bob's first turn the two neighbors of an uncolored vertex are colored with arcs of distance $1 - \varepsilon$ for an $\varepsilon > 0$ and $\varepsilon \to 0$. Without loss of generality assume that $dist(f(v_2), f(v_4)) = 1 - \varepsilon$. Then v_3 cannot not be assigned to an arc between $f(v_2)$ and $f(v_4)$ in the clockwise direction because $(v_2, v_3), (v_3, v_4) \in E$.



It is clear that for a graph G the ordinary circular chromatic number $\chi_c(G)$ is less than or equal to $\gamma_c(G)$ because Alice and Bob are competitive. For the trivial upper bound of the circular game chromatic number we consider the coloring of the vertex v and its neighbors $N(v)$, where v is a vertex with maximum degree $\Delta(G)$. Let $d(v) = \Delta(G) = n$ with $N(v) = \{v_1, ..., v_n\}$ and assume that all vertices from $N(v)$ are colored by Bob but v is uncolored yet. Without loss of generality assume that the vertices $\{v_1, ..., v_n\}$ have been assigned to arcs on C^r in the clockwise direction. If it holds that $0 < dist(f(v_i), f(v_{i+1})) < 1$ and $i \in \{1, ..., n-1\}$, then v cannot be colored with an arc between $f(v_i)$ and $f(v_{i+1})$ for $i \in \{1, ..., n-1\}$. Hence, we obtain that

$$\chi_c(G) \leq \gamma_c(G) \leq 1 + \Delta(G) + (\Delta(G) - 1) = 2\Delta(G).$$

One could conjecture that $1 + \Delta(G) + (\Delta(G) - 1)(1 - \varepsilon)$ for an $\varepsilon > 0$ is sufficient for coloring $N(v) \cup \{v\}$. Assume that a circle C^r with $r = 1 + \Delta(G) + (\Delta(G) - 1)(1 - \varepsilon)$ is given. If Bob colors $\{v_1, ..., v_n\}$ with arcs of distance $1 - \frac{\varepsilon}{n}$ with v uncolored yet, then $dist(f(v_i), f(v_{i+1})) = 1 - \frac{\varepsilon}{n}$ for $i \in \{1, ..., n-1\}$ and

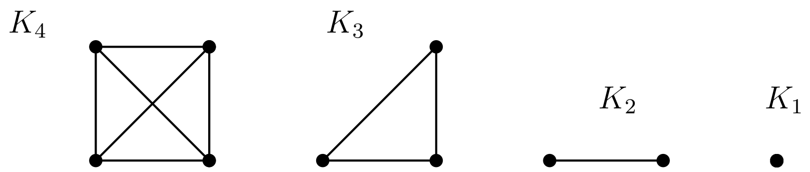
$$\begin{aligned}
dist(f(v_n), f(v_1)) &= \Big(1 + n + (n-1)(1-\varepsilon)\Big) - \Big(n + (n-1)(1 - \frac{\varepsilon}{n})\Big) \\
&= 1 + (n-1)(1-\varepsilon) - (n-1)(1 - \frac{\varepsilon}{n}) \\
&= 1 + (n-1)(1 - \varepsilon - 1 + \frac{\varepsilon}{n}) \\
&= 1 + (n-1)(\frac{\varepsilon}{n} - \varepsilon) \\
&< 1.
\end{aligned}$$

This implies that v cannot be colored on C^r. Thus, Bob wins.

2.1 The Circular Game Chromatic Number of Complete Graphs

In order to determine the circular game chromatic number of complete graphs we need to consider two cases. For K_n being the complete graph on n vertices we will distinguish between n odd and even. It is to be expected that by the structure of the K_n both players are indifferent which vertex to color when they take turns because none of the assigned arcs are allowed to overlap. However, we will work out strategies for both players how to place the corresponding arcs of the chosen vertices.

Definition 2.1.1. A graph $G = (V, E)$ is *complete* if every two vertices are adjacent to each other. K_n denotes the complete graph on n vertices.



Proposition 2.1.2. *Let K_n be a complete graph. Then*

$$\gamma_c(K_n) \leq \begin{cases} n + \lfloor \frac{n}{2} \rfloor & \text{for } n \text{ odd,} \\ n + \frac{n}{2} - 1 & \text{for } n \text{ even.} \end{cases}$$

Proof. We work out a strategy for Alice and determine the required circumference r on a circle C^r for which Alice wins, if she applies this strategy.

Alice's strategy: Initially, Alice colors an arbitrary vertex with an arbitrary arc on C^r. Then throughout the game she proceeds as follows. Let $v \in V$ be an uncolored vertex and $\{f_0, ..., f_{m-1}\}$ be the set of the assigned arcs on C^r in the clockwise direction.

- Assume that there exist two arcs f_i and f_{i+1} for $i \in \{0, ..., m-1\}$ such that $dist(f_{i \bmod m}, f_{i+1 \bmod m}) \geq 1$. Then Alice colors v with an arc between f_i and f_{i+1} such that either $dist(f_i, f(v)) = 0$ or $dist(f_{i+1}, f(v)) = 0$.
- Otherwise, if $dist(f_{i \bmod m}, f_{i+1 \bmod m}) < 1$ for all $i \in \{0, ..., m-1\}$, Bob wins the game by the assumption that K_n is complete.

Bob's strategy: Each time Bob takes turn he colors an arbitrary vertex with an arc of distance $1 - \varepsilon$ for an $\varepsilon > 0$ to an already placed arc.

Let n be odd. Then Alice finishes the game, if possible, because she colored first. Thus, during the game she colors $\lceil \frac{n}{2} \rceil$ vertices, while Bob colors $\lfloor \frac{n}{2} \rfloor$ vertices. The worst case for Alice occurs, if Bob applies his strategy above, which implies that he destroys $\lfloor \frac{n}{2} \rfloor$ additional arcs. Considering these facts we can conclude that for n odd it holds that

$$\gamma_c(K_n) \leq \left\lceil \frac{n}{2} \right\rceil + 2 \cdot \left\lfloor \frac{n}{2} \right\rfloor = n + \left\lfloor \frac{n}{2} \right\rfloor.$$

Let n be even. In this case Bob finishes the game; in particular, throughout the game both color $\frac{n}{2}$ vertices, respectively. Assume that $n-1$ vertices are colored and it is Bob's turn. Then due to the respective strategies Alice colored $\frac{n}{2}$ and Bob $\frac{n}{2}-1$ vertices, while destroying $\frac{n}{2}-1$ additional arcs. Obviously, for coloring the last vertex an arc of length 1 suffices such that Bob is forced to color with this arc. Thus, we can conclude that for n even it holds that

$$\gamma_c(K_n) \leq n + \frac{n}{2} - 1.$$

□

2.2 The Circular Game Chromatic Number of Cycles

In this chapter we restrict our attention to the circular game chromatic number of cycles. We work out a strategy for Bob and prove that $\gamma_c(G) = 4$ independent of which strategy Alice applies.

Definition 2.2.1. A *path* is a graph $P_n = (V, E)$ with

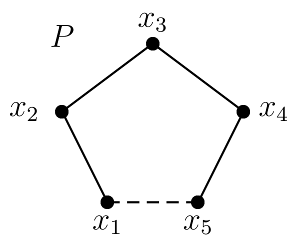
$$V(P_n) = \{x_1, x_2, ..., x_n\} \quad \text{and} \quad E(P_n) = \{(x_1, x_2), (x_2, x_3), ..., (x_{n-1}, x_n)\},$$

where the vertices x_i are distinct. The vertices x_1 and x_n are called the *endpoints* of P_n. For $x_1 = x_n$ the graph is called a *cycle* on n vertices and is denoted by C_n, where

$$E(C_n) = \{(x_1, x_2), (x_1, x_2), ..., (x_{n-1}, x_1)\}.$$

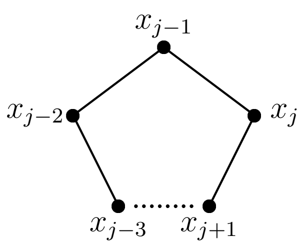
The number of the edges of a path or a cycle is its *length*. For $x, y \in V$ the *distance between u and v* is defined as the number of the edges of the shortest path between u and v and is denoted by $d_{(u,v)}$.

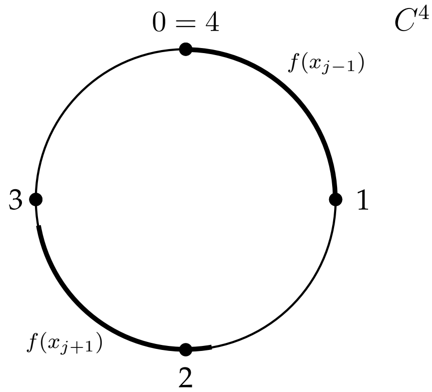
Example: Consider the path P_5 with $E(P_n) = \{(x_1, x_2), (x_2, x_3), (x_3, x_4), (x_4, x_5)\}$. Then $P_5 \cup (x_5, x_1)$ is a cycle.



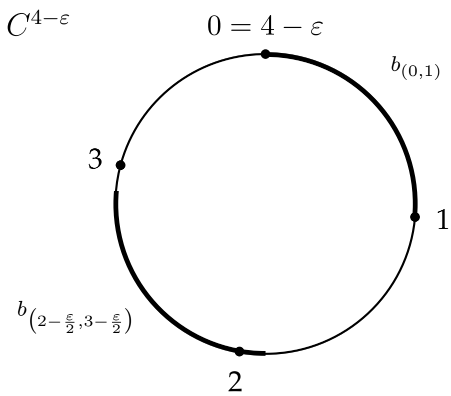
Proposition 2.2.2. *Let $C_n = (V, E)$ be a cycle. Then $\gamma_c(C_n) = 4$.*

Proof. Suppose a circle C^r with circumference $r = 4$ is given and let $V = \{x_0, x_1, ..., x_{n-1}\}$ and $E = \{(x_{i \bmod n}, x_{i+1 \bmod n}) \mid i \in \{0, ..., n-1\}\}$. Since each vertex has degree 2, during the game each uncolored vertex has at most two colored neighbors. Thus, it is sufficient to consider the coloring of an arbitrary path $\{x_{j-1 \bmod n}, x_{j \bmod n}, x_{j+1 \bmod n}\}$ for $j \in \{1, ..., n\}$ on C_n. Without loss of generality assume that Bob has colored the vertices x_{j-1} and x_{j+1} such that $dist(f(x_{j-1}), f(x_{j+1})) = 1 - \varepsilon$ for an $\varepsilon > 0$. This coloring illustrates the worst case that can occur, since the arc of length $1 - \varepsilon$ between $f(x_{j-1})$ and $f(x_{j+1})$ becomes useless for x_j. Nevertheless, since $r = 4$ there exists an arc between $f(x_{j+1})$ and $f(x_{j-1})$ of length 1 with which x_j can be colored. Hence, $\gamma_c(C_n) \leq 4$.





However, a circle with less circumference is not sufficient. Let $r = 4 - \varepsilon$. Without loss of generality assume that Alice colors x_{j-1} with the arc $b_{(0,1)}$ in her first turn. If Bob colors x_{j+1} with the arc $b_{(2-\frac{\varepsilon}{2}, 3-\frac{\varepsilon}{2})}$, a feasible coloring of x_j is not possible anymore because $l(b_{(3-\frac{\varepsilon}{2}, 4-\varepsilon)}) < 1$ and $l(b_{(1, 2-\frac{\varepsilon}{2})}) < 1$. Hence, Bob wins the game. Thus, $\gamma_c(G) = 4$.



□

2.3 The Circular Game Chromatic Number of Complete Multipartite Graphs

It is our purpose to study the circular game chromatic number for complete multipartite graphs. First, we give an upper bound for the circular game chromatic number of a complete multipartite graph $K_{s_1,...,s_n}$ for the case that $s_i \geq 4$ for all $i \in \{1, ..., n\}$. Afterwards, we assume that there exists at least one independent set with 3 vertices and prove a modified winning strategy for Alice on a circle with less circumference. Afterwards in section 2.3.1 we turn our attention to complete bipartite graphs minus a perfect matching M and show that $\gamma_c(K_{m,m}) \leq \gamma_c(K_{m,m} - M)$.

Definition 2.3.1. Let $n \geq 2$ be an integer. A graph $G = (V, E)$ is called *n-partite* or *multipartite* if V admits a partition into n classes $S_1, ..., S_n$ such that every edge has its ends in different classes. Vertices in the same partition are not allowed to be adjacent. We denote a multipartite graph by $M_{s_1,...,s_n}$ with independent sets $S_1, S_2, ..., S_n$ where $s_i = |S_i|$ for $i = 1, ..., n$. Instead of a 2-partite graph one usually says *bipartite graph*.

For the case that each two vertices from S_i and S_j are adjacent we call the graph a *complete multipartite graph* and denote it by $K_{s_1,...,s_n}$.

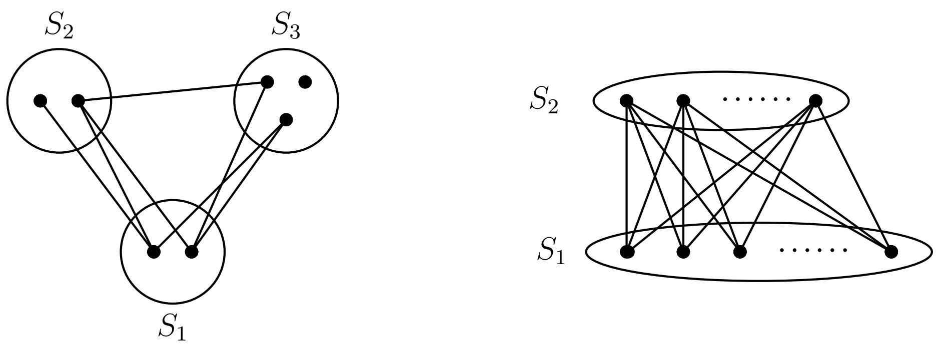


FIGURE: The multipartite graph M_{s_1,s_2,s_3} with independent sets $\{S_1, S_2, S_3\}$ for $s_1 = s_2 = 2$ and $s_3 = 3$ and the complete bipartite graph K_{s_1,s_2} with independent sets $\{S_1, S_2\}$ for $s_1 \geq s_2$.

By the structure of a complete multipartite graph, it is clear that once a vertex from an independent set S_i is colored, the remaining uncolored vertices of S_i can be colored with the same arc. Then we say that S_i is *save* from Alice's view.

Proposition 2.3.2. *Let $K_{s_1,...,s_n} = (V, E)$ be a complete multipartite graph with $s_i \geq 4$ for all $i \in \{1, ..., n\}$. Then it holds $\gamma_c(K_{s_1,...,s_n}) \leq 3n - 2$.*

Proof. The procedure is to work out a strategy for Alice and to determine the required circle C^r which guarantees her victory.
Alice's Strategy: Let $C \subseteq V$ be the set of all colored vertices of $K_{s_1,...,s_n}$. Initially, Alice colors an arbitrary vertex. Then each time she takes turn she goes ahead as follows. Let $\{f_0, ..., f_m\}$ be the set of all assigned arcs on C^r in the clockwise direction. Assume that there exists a j_0 such that $S_{j_0} \cap C = \emptyset$.

- If there exist two arcs $f_{i \bmod m}$ and $f_{i+1 \bmod m}$ such that $dist(f_{i \bmod m}, f_{i+1 \bmod m}) \geq 1$ for $i \in \{0, ..., m\}$, then Alice colors an arbitrary vertex v from S_{j_0} with an arc between $f_{i \bmod m}$ and $f_{i+1 \bmod m}$ such that either $dist(f_{i \bmod m}, f(v)) = 0$ or $dist(f_{i+1 \bmod m}, f(v)) = 0$.

- If $dist(f_{i \bmod m}, f_{i+1 \bmod m}) < 1$ for all $i \in \{0, ..., m\}$, then by the structure of a complete multipartite graph Alice looses the game because the vertices of S_{j_0} cannot be colored with a feasible arc.

Let $S_j \cap C \neq \emptyset$ for all $j \in \{1, ..., n\}$. Then Alice chooses the vertices to color at random such that vertices from the same independent set are colored with the same arc.

What is left is the worst case scenario that can occur, which is the following: Let $\{f_0, ..., f_m\}$ be the set of all assigned arcs on C^r and let $S' = \{S_1, ..., S_k\}$ for $1 \leq k \leq n$ such that $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., k\}$. Suppose that there exists at least one independent set S_l such that $S_l \cap C = \emptyset$. Otherwise, if $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., n\}$, then by the structure of the graph the coloring of the remaining uncolored vertices is fixed.

- Assume that there exist two arcs $f_{i \bmod m}$ and $f_{i+1 \bmod m}$ for $i \in \{0, ..., m\}$ such that $dist(f_{i \bmod m}, f_{i+1 \bmod m}) \geq 2$. Then Bob colors an arbitrary vertex from S' with an arc of distance $1 - \varepsilon$ to $f_{i \bmod m}$ for an $\varepsilon > 0$.

- Assume that $dist(f_{i \bmod m}, f_{i+1 \bmod m}) < 2$ for all $i \in \{0, ..., m\}$ and that there exists one $j \in \{0, ..., m\}$ such that $1 \leq dist(f_{j \bmod m}, f_{j+1 \bmod m}) < 2$. Then Bob colors an arbitrary vertex from S' with an arc of distance 0 to $f_{j \bmod m}$ or to $f_{j+1 \bmod m}$.

- Assume that $dist(f_{i \bmod m}, f_{i+1 \bmod m}) < 1$ for all $i \in \{0, ..., m\}$. Then Bob wins the game because the vertices of S_l cannot be colored with a feasible arc.

Obviously, if Bob colored in an independent set that does not contain colored vertices, a smaller circle would be sufficient for Alice's victory. Moreover, assigning the arcs with distance of $1 - \varepsilon$ is clearly the maximum he is able to destroy in each move.

Alice's strategy and Bob's worst case strategy imply that n distinct unit length arcs are assigned by Alice and $n - 1$ by Bob, whereas he additionally destroys $n - 1$ further arcs. Hence, we get the following calculation:

$$\gamma_c(K_{s_1,...,s_n}) \leq n + 2(n - 1) = 3n - 2.$$

□

Corollary 2.3.3. *The circular game chromatic number for bipartite graphs is* 4. □

We have been working under the assumption that every independent set contains at least 4 vertices. It is worth pointing out that less circumference than $3n-2$ suffices if we admit that there exists at least one independent set S_{i_0} with $s_{i_0} = 3$. Assume a circle with circumference $r = 3n - 3$ is given. Then a slight change in Alice's strategy is required in order to achieve a feasible coloring on C^r for $r = 3n - 3$.
Alice's modified strategy: Without loss of generality assume that $s_1 = 3$ with $S_1 = \{v_{1_1}, v_{1_2}, v_{1_3}\}$. Initially, Alice colors vertex v_{1_1} with an arbitrary arc. By his worst case strategy Bob colors vertex v_{1_2} such that $dist(f(v_{1_1}), f(v_{1_2})) = 1 - \varepsilon$ for an $\varepsilon > 0$. In contrast to her strategy in proposition 2.3.2 Alice does not go ahead with S_i for $i \in \{2, ..., n\}$ but colors the remaining uncolored vertex of S_1 such that v_{1_3} is colored with the arc $f(v_{1_1})$ or $f(v_{1_2})$. Since $s_1 = 3$, Bob is forced to jump into $\{S_2, ..., S_n\}$ and to color an uncolored independent set first, which would be save from Alice's view. In particular, the required circumference decreases by 1. However, they proceed the game using the same strategies as in proposition 2.3.2. This implies that throughout the game Alice and Bob assign $n - 1$ distinct unit length arcs, respectively, whereas Bob additionally destroys $n - 1$ further arcs. Hence, we can conclude that a circle with circumference $r = (n - 1) + 2(n - 1) = 3n - 3$ suffices for achieving a feasible coloring of the graph.

2.3.1 The Circular Game Chromatic Number of Complete Bipartite Graphs without a Perfect Matching

One may conjecture that the circular game chromatic number decreases if we reduce the number of the edges of a graph. However, it is well known, that $3 = \gamma(K_{m,m}) \leq \gamma(K_{m,m} - M) = m$, where the $K_{m,m}$ is a complete bipartite graph and M is a perfect matching of $K_{m,m}$. We found out that it also holds for the circular version of the game, such that for $m \geq 3$

$$4 = \gamma_c(K_{m,m}) \leq \gamma_c(K_{m,m} - M).$$

Moreover, we will show that $\gamma_c(K_{m,m} - M) > m$.

Due to a corollary of the so-called *marriage theorem* of Hall, [9], every k-regular bipartite graph contains a perfect matching. Thus, since the $K_{m,m}$ is m-regular, it contains a perfect matching. In the following we will consider the vertex set $V(K_{m,m})$ as the disjoint union of the two distinct independent sets, denoted by A and B, where $A = \{a_1, a_2, ..., a_m\}$ and $B = \{b_1, b_2, ..., b_m\}$. Obviously, $M = \{(a_1, b_1), (a_2, b_2), ..., (a_m, b_m)\}$ is a perfect matching. See the following figure.

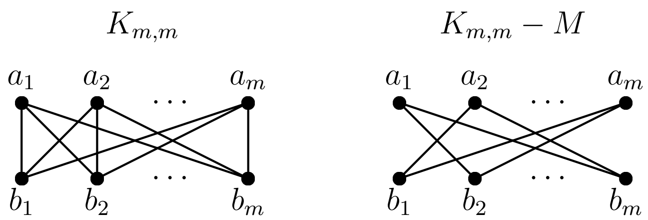


FIGURE: $K_{m,m}$ and $K_{m,m} - M$ with $M = \{(a_1, b_1), (a_2, b_2), ..., (a_m, b_m)\}$.

Remark 2.3.4. Obviously, the coloring of the entire graph is trivial, if there exists $i_0 \in \{1, ..., m\}$ such that $f(a_{i_0}) \cap f(b_{i_0}) = \emptyset$ because of the following consideration: Since $(a_{i_0}, b_j) \in E(K_{m,m} - M)$ for $i_0 \neq j$, none of the remaining uncolored vertices from $\{b_1, ..., b_m\}$ can be colored with an arc that is allowed to overlap with $f(a_{i_0})$. Thus, the coloring of the remaining uncolored vertices $\{a_1, ..., a_m\}$ is fixed, since they can be colored with $f(a_{i_0})$ The same conclusion can be drawn for the coloring of $\{b_1, ..., b_m\}$.

Proposition 2.3.5. *Let $K_{m,m} = (V, E)$ be a complete bipartite graph and M a perfect matching of $K_{m,m}$ for $m \geq 3$ and $m \in \mathbb{N}$. Then*

$$m < \gamma_c(K_{m,m} - M) \leq m + 2.$$

Proof. Let $M = \{(a_1, b_1), (a_2, b_2), ..., (a_m, b_m)\}$. We denote by $A = \{a_1, ..., a_m\}$ and $B = \{b_1, ..., b_m\}$ the two independent sets. For the purpose of determining the upper bound, we will work out a strategy for Alice and determine the required circumference for achieving a feasible coloring of the graph. Let U denote the set of uncolored vertices and W the set of colored vertices during the game. Obviously, $U = V$ and $W = \emptyset$ at the beginning of the game.

Alice's strategy:

- Initially, Alice colors an arbitrary vertex.
- Assume that there exists an $i \in \{1, ..., m\}$ with x_i being colored and y_i being uncolored for $x, y \in \{a, b\}$ and $x \neq y$. Alice colors y_i with an arc such that $f(x_i) \cap f(y_i) = \emptyset$.
- Assume there does not exist an $i \in \{1, ..., m\}$ with x_i being colored and y_i being uncolored for $x, y \in \{a, b\}$ and $x \neq y$ and let the last vertex colored by Bob be an element of $H \in \{A, B\}$. Further, assume $W \neq V$. Then Alice colors vertex $z \in H$ with an arc such that
  - $dist(f(z), f(q)) = 0$ for $q \in W$ and $q \in \{A, B\} \setminus H$,
  - if possible, $f(z) \cap f(z') \neq \emptyset$ for $z' \in H$.

According to Alice's strategy we give Bob's strategy and explain why it turns out to be the worst case scenario.

Bob's strategy:

(i) Assume $|U| \geq 4$. If Alice colors a vertex x_j for $x \in \{a, b\}$ and $j \in \{1, ...m\}$, then Bob colors y_j for $y \in \{a, b\}$ and $y \neq x$ such that $l\big(f(x_j) \cap f(y_j)\big) = \varepsilon$ and $dist(f(y_k), f(y_j)) = 1 - \varepsilon$ for a $y_k \in W$ and for an $\varepsilon > 0$ (according to Alice's strategy such an $f(y_k)$ exists).

(ii) Assume $|U| \leq 3$ and let x_j be the last vertex colored by Alice for $x \in \{a, b\}$. Bob colors vertex $x_{j'}$ such that $dist(f(x_{j'}), f(x_i)) = 1 - \varepsilon$ for an $x_i \in W$.

Further, we explain why Bob's strategy illustrated above is the worst case for Alice's strategy. Suppose Alice colors a_i and assume that $|U| \geq 4$.

- If Bob colored a vertex other than b_i, then by the remark 2.3.4 Alice would be able to fix the coloring in her next turn by coloring b_i such that $f(b_i) \cap f(a_i) = \emptyset$ holds.
- If Bob colored b_i with an arc $f(b_i)$ such that $f(b_i) \cap f(a_i) = \emptyset$, then again by 2.3.4 the coloring would be fixed.

Thus, the obvious worst case is to color b_i such that $l\big(f(a_i) \cap f(b_i)\big) = \varepsilon$ and $dist(f(b_k), f(b_i)) = 1 - \varepsilon$ for a $b_k \in W$. The existence of such a b_k is guaranteed since Alice colors vertices with arcs next to already placed arcs. So far, by the strategies of both players we can conclude that $m - 1$ unit length arcs suffice to color the first $2m - 4$ vertices.

Further assume $|U| \leq 3$. Without loss of generality we consider the procedure of the game where

- the vertices a_i and b_i for all $i \in \{1, ..., m - 2\}$ have been colored
- Alice started the game coloring a_1 and
- $\{a_{m-1}, a_m, b_{m-1}, b_m\}$ are uncolored yet.

Since the number of the already colored vertices is even, Alice is the next one to color. In particular, due to the respective strategies of the players, a_{m-2} is the last colored vertex, if m is even, and b_{m-2} is the last colored vertex, if m is odd. Without loss of generality assume that m is even and Bob has colored a_{m-2} in his last turn.

Figures (a) and (b) demonstrate the coloring of $\{a_1, b_1, ..., a_{m-2}, b_{m-2}\}$, where $M = \{(a_1, b_1), (a_2, b_2), ..., (a_m, b_m)\}$.

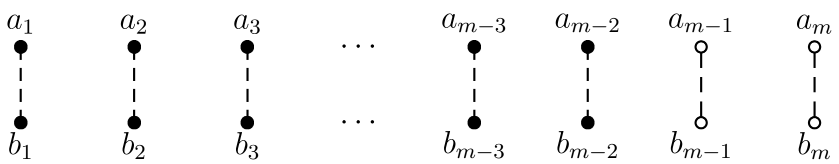


FIGURE (a): The dashed lines indicate the edges in the matching $M = \{(a_1, b_1), (a_2, b_2), ..., (a_m, b_m)\}$

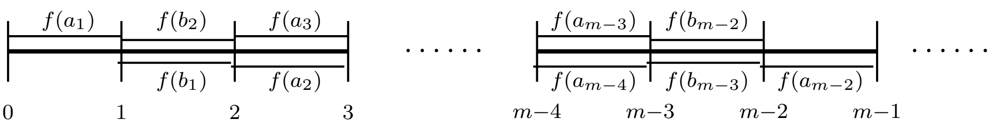


FIGURE (b): The coloring of the vertices $\{a_1, b_1, ..., a_{m-2}, b_{m-2}\}$

Then due to her strategy, Alice colors a_{m-1}. At this point of the game we can conclude that in the independent set A Bob has colored $\frac{m-2}{2}$ vertices, while destroying $\frac{m-2}{2}$ arcs of length $1-\varepsilon$. In B Bob also has colored $\frac{m-2}{2}$ vertices but since he started coloring in B (since Alice started coloring a_1, Bob colored b_1), he has destroyed $\frac{m-2}{2}-1$ arcs of length $1-\varepsilon$. Thus, obviously it makes sense for Bob to color a_m instead of b_{m-1}. Thus, because of (ii), an additional circumference of 3 is required for coloring b_{m-1} and a_m, while another arc of length $1-\varepsilon$ is being destroyed.

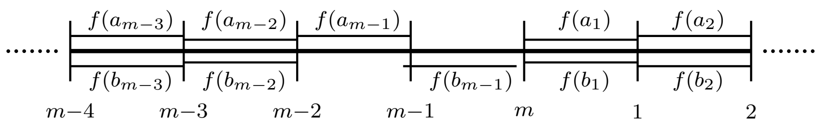
Finally, we can conclude that $\gamma_c(K_{m,m}-M) \leq (m-1)+3 = m+2$.

Further, we show that a circle with circumference m does not suffice to guarantee Alice's victory. In particular, we will give a winning strategy for Bob. In the following we call a *free interval of length* k an arc of length k on C^m on which there is no vertex colored.

- Assume that on the circle C^m there are at least two free intervals of length 2 or 3. Then Bob does the following: If Alice colors x_{i_0} where $x \in \{a,b\}$ and $i_0 \in \{1,...,m\}$, then Bob colors y_{i_0} where $y \in \{a,b\}$ and $y \neq x$ with $f(x_{i_0})$. Clearly, if Bob uses this strategy, at some point of the game there will exist only one free interval of length 2, respectively 3.

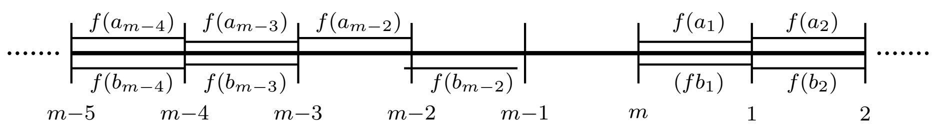
  If Alice colors such that there exists a free interval of length l or k where $2 < l < 3$, respectively, $1 < k < 2$, then she obviously loses the game, since this implies that on the circle C^m there exists a free interval of length $3-l$ or $2-k$ on which there cannot be colored any vertices. By Bob's strategy, Alice can win the game if and only if at the end there does not exist any free interval on C^m.

- Assume now that it is Alice's turn and that there is one free interval of length 2 and the other free intervals have length less than 2. Without loss of generality assume that the vertices $a_1, a_2, ..., a_{m-2}$ and $b_1, b_2, ..., b_{m-2}$ are colored. Further, we can assume that Alice colors a_{m-1}. Then Bob colors b_{m-1} such that $f(a_{m-1}) \cap f(b_{m-1}) = \varepsilon$ for an $\varepsilon > 0$ and $f(a_{m-1}) \neq f(b_{m-1})$. For vertex a_m there does not exist a feasible arc anymore and hence Bob wins the game.

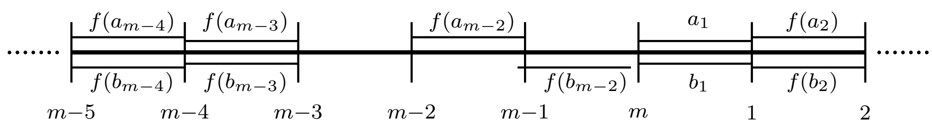


- Assume now that it is Alice's turn and there is one free interval of length 3 and the other free intervals have length less than 3. Without loss of generality assume that the vertices $a_1, a_2, ..., a_{m-3}$ and $b_1, b_2, ..., b_{m-3}$ are colored. Further we can assume that Alice colors a_{m-2}. Two cases are possible. Either Alice colors a_{m-2} such that a free interval of length 2 is left or two free intervals of length 1.

  Case 1: Assume Alice colors a_{m-2} such that a free interval of length 2 is produced. Then Bob colors b_{m-2} so that $f(a_{m-2}) \cap f(b_{m-2}) = \varepsilon$. After Alice's next move, Bob will obviously have the opportunity to color either b_{m-1} or b_m with $b_{(m-1,m)}$ such that he wins the game because for either a_{m-1} or a_m there does not exist a feasible arc.



  Case 2: Assume Alice colors a_{m-2} such that two free intervals of length 1 are produced. Then Bob colors b_{m-2} so that $f(a_{m-2}) \cap f(b_{m-2}) = \varepsilon$. After Alice's next move, Bob will obviously have the opportunity to color either b_{m-1} or b_m with $b_{(m-1,m)}$ or $b_{(m-3,m-2)}$ such that he wins the game because for either a_{m-1} or a_m there does not exist a feasible arc.



□

2.4 Circular Game Chromatic Number of Planar Graphs

Planar graphs received huge interest in the graph theoretic community. The definition is easy to explain. A graph is called *planar* iff it can be drawn on a plane without edge crossing. A lot of strong theorems have been proved on this topic. The most famous is the theorem of Kuratowski, which proves that a graph is planar if and only if it does not contain a K_5 or a $K_{3,3}$. A further essentially conjecture is the *Four Color Conjecture*, which claims that every planar graph can be colored with 4 colors. So far there does not exist an ordinary mathematical proof. However, 1977 K. Appel and W. Haken presented a proof of the four color theorem using computer sciences. Concerning the game chromatic number, there does not exist a precise result for planar graphs. However, a lot of efforts have been made to find sharp upper bounds. The best known is 17 worked out by Zhu in [25]. Our approach is to give an upper bound for the circular game chromatic number of planar graphs.

Kierstead bounded in [11] the marking game number of a graph G (see page 8) which restricts the game chromatic number in terms of a parameter $r(G)$, the rank of a graph G. In particular, he worked out a winning strategy for Alice for the marking game, the so called *activation strategy*, and proved that the game chromatic number of planar graphs is at most 18. Our approach is to extend the marking game and to introduce the circular marking game on graphs. In addition, we intend to determine the circular game chromatic number of planar graphs using techniques by Kierstead.

2.4.1 The Activation Strategy for the Circular Game

- We denote by $\Pi(G)$ the set of linear orderings on the vertices of G.
- For a linear ordering L we obtain the *orientation* G_L of G with respect to L by setting $E_L = \{(v,u) : \{v,u\} \in E \text{ and } v > u \text{ in } L\}$.
- Let $V^+_{G_L}(u) = \{v \in V | v < u\}$ and $V^-_{G_L}(u) = \{v \in V | v > u\}$ with $V^+_{G_L}[u] = V^+_{G_L}(u) \cup \{u\}$ and $V^-_{G_L}[u] = V^-_{G_L}(u) \cup \{u\}$.

- For a vertex $u \in V(G)$ we denote the outneighborhood of u in G_L by $N^+_{G_L}(u)$ and the inneighborhood of u in G_L by $N^-_{G_L}(u)$ with $N^+_{G_L}[u] = N^+_{G_L}(u) \cup \{u\}$ and $N^-_{G_L}[u] = N^-_{G_L}(u) \cup \{u\}$. The various degrees of u are denoted by $d^+_{G_L}(u) = |N^+_{G_L}(u)|$ and $d^-_{G_L}(u) = |N^-_{G_L}(u)|$. The *maximum outdegree* of G_L is denoted by $\Delta^+_{G_L}$ and the *maximum indegree* by $\Delta^-_{G_L}$.

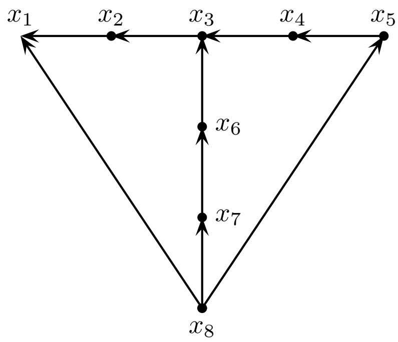


FIGURE: G_L with $E_L = \{(x_2, x_1), (x_3, x_2), (x_4, x_3), (x_5, x_4), (x_6, x_3), (x_7, x_6), (x_8, x_7), (x_8, x_1), (x_8, x_5)\}$

The Circular Marking Game

The *circular marking game* is played on a finite graph G by Alice and Bob with Alice playing first. In each move the players take turns choosing vertices from the shrinking set $U \in V(G)$ of unchosen vertices. This results in a linear ordering L of the vertices of G with $x < y$ iff x is chosen before y. Notice $L \in \Pi(G)$ where $\Pi(G)$ is the set of linear orderings on $V(G)$. The *c-score* of the game is equal to $1 + \Delta^+_{G_L} + (\Delta^+_{G_L} - 1) = 2\Delta^+_{G_L}$. Alice wins, if the c-score is at most a given integer t; otherwise, Bob wins.

The circular marking game number $col^c_g(G)$ of a graph $G = (V, E)$ is the least integer t such that Alice has a winning strategy for the circular marking game, that is $2\Delta^+_{G_L} \leq t$.

Remark 2.4.1. Note that the circular marking game differs from the ordinary marking game in terms of the score of the game. While the score of the ordinary marking game is $\Delta^+_{G_L} + 1$, where $\Delta^+_{G_L}$ is defined as the maximum number of marked neighbors of an unmarked vertex, it turns out to be the worst case for the coloring game. The idea behind, is that the neighbors $N(v)$ of a vertex v

get distinct colors. For bounding the circular game chromatic number we have to go a step further and to consider the positions of the corresponding arcs of $N(v)$ on the cycle C^r, while assuming that Bob colored them. Thus, another arc of length $\Delta^+_{G_L} - 1$ is required, if the worst case occurs.

Proposition 2.4.2. *For a graph $G = (V, E)$ it holds $\gamma_c(G) \leq col_g^c(G)$.*

Proof. Suppose we have a circle with circumference $col_g^c(G)$. If Alice follows her optimal strategy of the circular marking game and colors the vertex by First-Fit, which is to be chosen, she wins also the circular coloring game. $\square$

The Activation Strategy $S(L, G)$ and the Rank of G

Kierstead introduced the *activation strategy* which restricts $\Delta^+_{G_L}$ if Alice applies this strategy. For this purpose, he defined the so called *rank* of a graph G. For the sake of completeness we will give below this strategy.

Fix a graph G and a linear ordering $L \in \Pi(G)$. Let $A \subset V$ be the set of *active* vertices with $A := \emptyset$ at the beginning. Furthermore let U denote the set of unchosen vertices. Alice starts the game by activating the least vertex from L and chooses it. After Bob has chosen any vertex b from U Alice does the following:

Strategy $S(L, G)$

- `x` $:=$ `b;`
- `while` $\mathtt{x} \notin \mathtt{A}$ `do` $\mathtt{A} := \mathtt{A} \cup \{\mathtt{x}\}$`;` $\mathtt{s(x)} := \min_{\mathrm{L}} \mathtt{N}^+[\mathtt{x}] \cap (\mathtt{U} \cup \{\mathtt{b}\})$`;` `x` $:=$ `s(x) od;`
- `if` $\mathtt{x} \neq \mathtt{b}$ `then choose x`

  `else` $\mathtt{y} := \min_{\mathrm{L}} \mathtt{U}$`; if` $\mathtt{y} \notin \mathtt{A}$ `then` $\mathtt{A} := \mathtt{A} \cup \{\mathtt{y}\}$ `fi; choose y fi;`

The Rank of a Graph

Let $A, B \subset V(G_L)$. A matching M is a matching from A to B if M saturates A and $B \setminus A$ contains a cover of M. As mentioned above $\Delta^+_{G_L}$ is bounded in terms of the following parameters. For $u \in V(G)$ the *matching number* $m(u, L, G)$ of u with respect to L is defined as the size of the largest set $Z \subset N^-[u]$ such that there exists a partition $\{X, Y\}$ of Z and there exist matchings M from $X \subset N^-[u]$ to $V^+(u)$ and N from $Y \subset N^-(u)$ to $V^+[u]$. The *rank* $r(L, G)$ of G with respect to L and $rank$ $r(G)$ of G are defined by

$$r(u, L, G) := d^+_{G_L}(u) + m(u, L, G)$$

$$r(L, G) := \max_{u \in V} r(u, L, G)$$

$$r(G) := \min_{L \in \Pi(G)} r(L, G)$$

For our case we shall extend this definition and define *the circular rank* of a graph G:

The Circular Rank of a Graph

$$r^c(u, L, G) := 2(d^+_{G_L}(u) + m(u, L, G)) - 1$$

$$r^c(L, G) := \max_{u \in V} r^c(u, L, G)$$

$$r^c(G) := \min_{L \in \Pi(G)} r^c(L, G)$$

The next proposition shows the relation between the circular game marking number and the circular rank of a graph G.

Proposition 2.4.3. *For any graph $G = (V, E)$ and ordering $L \in \Pi(G)$, if Alice uses the strategy $S(L, G)$ in order to play the circular marking game on G, then the c-score will be at most $1 + r^c(L, G)$. In particular, $col^c_g(G) \leq 1 + r^c(G)$.*

Proof. Suppose that Alice applies strategy $S(L, G)$ for the circular marking game on G. Since every vertex chosen by Bob immediately becomes active and any vertex chosen by Alice is already active, it remains to show that at any time t any unchosen vertex u is adjacent to at most $d^+_{G_L}(u) + m(u, L, G)$ active vertices instead of chosen vertices, that is $|N(u) \cap A| \leq d^+_{G_L}(u) + m(u, L, G)$.

It is obvious that $|N(u) \cap A| \leq d^+_{G_L}(u) + |N^-(u) \cap A|$, and since Kierstead proved in [11] that $|N^-(u) \cap A| \leq m(u, L, G)$, we can conclude that

$$|N(u) \cap A| + |N(u) \cap A| - 1 \leq$$

$$d^+_{G_L}(u) + |N^-(u) \cap A| + d^+_{G_L}(u) + |N^-(u) \cap A| - 1 \leq$$

$$d^+_{G_L}(u) + m(u, L, G) + d^+_{G_L}(u) + m(u, L, G) - 1 = r^c(u, L, G)$$

□

In proposition 2.4.3 we proved that $col^c_g(G)$ is bounded by the circular rank. In order to determine the circular game chromatic number of a graph G or a class of graphs one can determine the circular rank by the proposition 2.4.2. In the following we will use this result to bound γ_c for planar graphs.

The Circular Game Chromatic Number of Planar Graphs

Corollary 2.4.4. *If G is a planar graph, then $col^c_g(G) \leq 34$.*

Proof. Let $G = (V, E)$ be a planar graph. Kierstead proved in [11] that at any time for an unchosen vertex $u \in V(G)$ it holds:

$$d^+_{G_L}(u) + m(u, L, G) \leq 17.$$

Using this result, we found out that for the vertex $u \in V(G)$

$$r^c(u, L, G) \leq 33$$

holds. It follows for the circular marking game number of a planar graph G that

$$col^c_g(G) \leq 34.$$

□

In this manner, according to [11], we are able to estimate easily the circular game chromatic number for trees.

Definition 2.4.5. A graph that does not contain any cycles is called a *forest*. A connected forest is called a *tree* and is denoted by $T = (V, E)$.

Remark 2.4.6. If $T = (V, E)$ is a tree, then $col_g^c(G) \leq 6$.

Proof. Let L be an ordering of V so that $|N_L^+(v)| \leq 1$ for every $v \in V(G)$ (since T is a tree such a linear ordering can be easily found). Then obviously $col_g^c(G) \leq 6$. ☐

2.5 The Circular Game Chromatic Number of Cactuses

Definition 2.5.1. A graph $K = (V, E)$ is a *cactus* if any two cycles of K have at most one common vertex.

The aim of this section is to determine the circular game chromatic number for the class of cactuses by using techniques of Sidorowicz in [19]. Sidorowicz showed that the game chromatic number for the class of cactuses is 5. First, she proved that 5 is an upper bound, using lemma 2.5.3 proved by Zhu in [26], by showing that $col_g(\mathcal{C}) \leq 5$ for $\mathcal{C}$ being the class of cactuses. Further, she proved that there exists a cactus for which Alice has no winning strategy if four colors are given. In particular, Sidorowicz showed the following lemma.

Lemma 2.5.2. *If $K = (V, E)$ is a cactus, then there is a matching M such that $K - M$ is an acyclic graph.*

Note that an *acyclic* graph is a graph which does not contain any cycles.

Hence, we can decompose a cactus $K = (V, E)$ into a forest F and a matching M with $K = F \cup M$. Using the lemma below Sidorowicz was able to give an upper bound.

Lemma 2.5.3. *Suppose $G = (V, E)$ and $E = E_1 \cup E_2$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Then $col_g(G) \leq col_g(G_1) + \Delta(G_2)$.*

Since $col_g(F) \leq 4$ for an acyclic graph, she concluded for a cactus $K =$

(V, E) that $col_g(K) \leq 4 + 1 = 5$ and hence $\gamma(K) \leq 5$. As mentioned above she proved further the existence of a cactus with game chromatic number greater or equal to 5 and finally conclude that the game chromatic number for the class of cactuses is equal to 5. We will use these techniques for determining the circular game chromatic number for the class of cactuses. Further we will call a vertex $v \in V(G)$ for a graph G *pendant* if $d(v) = 1$. First, we give the circular version of Zhu's lemma.

Lemma 2.5.4. *Suppose $G = (V, E)$ and $E = E_1 \cup E_2$. Let $G_1 = (V, E_1)$ and $G_2 = (V, E_2)$. Then $col_g^c(G) \leq col_g^c(G_1) + 2\Delta(G_2)$.*

Proof. Suppose $t = col_g^c(G_1) + 2\Delta(G_2)$. If Alice applies her optimal strategy of G_1 to play the marking game on G, then she wins:

Assume Alice applies her optimal strategy of G_1 and assume that $L \in \Pi(G_1)$ is a linear ordering of V which is produced through the game. For this linear order L we can conclude:

$$\Delta^+_{G_{1_L}} + 1 + (\Delta^+_{G_{1_L}} - 1) \leq col_g^c(G_1) \Rightarrow$$

$$\Delta^+_{G_{1_L}} + 1 + (\Delta^+_{G_{1_L}} - 1) + 2\Delta(G_2) \leq col_g^c(G_1) + 2\Delta(G_2).$$

Obviously

$$col_g^c(G) \leq \Delta^+_{G_{1_L}} + 1 + (\Delta^+_{G_{1_L}} - 1) + 2\Delta(G_2)$$

holds and hence

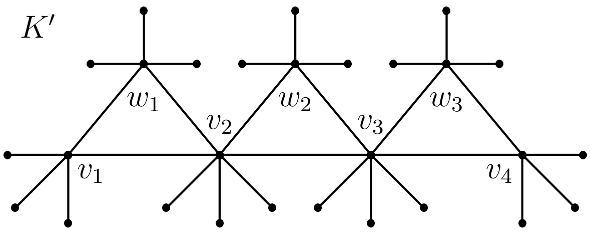
$$col_g^c(G) \leq col_g^c(G_1) + 2\Delta(G_2).$$

□

According to lemma 2.5.4 we can conclude for the class of cactuses:

Corollary 2.5.5. *Let $\mathcal{C}$ be the class of cactuses. Then $col_g^c(\mathcal{C}) \leq 8$.* □

Further, we prove that there exists a cactus with circular game chromatic number greater or equal to 5. In particular, we will consider the cactus below (by Sidorowicz [19]) which we will denote by K'.



Lemma 2.5.6. *For the cactus K' it holds $\gamma_c(K') \geq 5$.*

Proof. Suppose a circle with circumference $5 - \varepsilon$ for an $\varepsilon > 0$ is given. Assume Alice starts the game coloring v_1 with α. Without loss of generality let $\alpha = b_{(0,1)}$. Then Bob colors v_3 with $b_{(2-\frac{\varepsilon}{4}, 3-\frac{\varepsilon}{4})}$. This forces Alice to color v_2 in her next turn. She must color it with an arc between $3 - \frac{\varepsilon}{4}$ and $5 - \varepsilon$. However, Alice colors v_2, Bob easily can color a pendant vertex of w_1 or w_2 such that a circle of circumference $5 - \varepsilon$ does not suffice for coloring w_1 or w_2, respectively. If Alice starts coloring another vertex, then Bob colors a vertex of distance 2 using the strategy above and wins the game. □

Chapter 3

The Two-Person Game on Weighted Graphs

In this chapter we will introduce a modification of Bodlaender's two-person game by taking vertex-weights into account. Thus, throughout the chapter we will consider a weighted graph (G, w) such that to the vertices of (G, w) are assigned positive integers, the so called *vertex-weights*.

While for coloring a graph without vertex-weights the coloring mapping $c : V \to \mathbb{N}$ must satisfy the condition $c(u) \neq c(v)$ for $(u, v) \in E$, for a feasible coloring of a weighted graph an additional condition for c must hold:

$$c(v) \geq w(v),$$

where $w(v)$ is the vertex-weight of the vertex $v \in V$. The difficulty of the new concept is that for the strategies of the respective players, besides the structure of the graph also the given distribution of the vertex-weights is decisive. We wish to investigate the two-person game on weighted graphs and to discuss the game chromatic number $\gamma(G, w)$ for the class of complete graphs, complete multipartite graphs, cycles, trees and planar graphs. For the purpose of bounding $\gamma(G, w)$, we will extend the marking game for our case and define the *weighted marking game number*, denoted by $col_g^w(G)$, which turns out to be an upper bound of $\gamma(G, w)$. For determining the weighted marking game number we will modify in section 3.6 Kierstead's activation strategy, which we already pointed out on page 28.

It is to be expected that $\gamma(G, w) \geq \gamma(G)$ because $c(v) \geq w(v)$ and $w(v) \geq 1$, $v \in V$, must hold for a feasible coloring of (G, w). However, in section 3.3 we will consider a distribution of vertex-weights by means of a cycle such that $\gamma(G, w) < \gamma(G)$ holds. In section 3.4 we will look more closely at this surprising result and construct graphs with property $\gamma(G, w) < \gamma(G)$.

The Game Chromatic Number of Weighted Graphs

We denote the game chromatic number of a weighted graph (G, w) by $\gamma(G, w)$ and define it as follows:

Definition 3.0.7. Let $(G, w) = (V, E, w)$ be a weighted graph. The *game chromatic number* $\gamma(G, w)$ of (G, w) is equal to the smallest amount of given colors, such that there exists a winning strategy for Alice for the coloring game.

Obviously, if the mapping w takes the constant value 1, then $\gamma(G) = \gamma(G, 1)$.

For the remainder of this chapter we define for a vertex set $U = \{u_1, ..., u_n\}$

$$w_{\max}(U) := \max\{w(u_i) \mid u_i \in U\} \text{ and}$$

$$w_{\min}(U) := \min\{w(u_j) \mid u_j \in U\}$$

as the greatest and the least weight, respectively, that is being assigned to a vertex by the mapping w. For simplicity, we will write $w_{\max}(G, w)$ instead of $w_{\max}(V(G, w))$, if we consider the greatest weight which is assigned to a vertex from the entire vertex set $V(G, w)$. Further,

$$w_{\max_2}(G, w) := \max\{w(u_i) \mid w(u_i) < w_{\max}(G, w)\}$$

is the second greatest vertex-weight of (G, w).

3.1 The Game Chromatic Number of Weighted Complete Graphs

In this section we will analyze the two-person game on weighted complete graphs (K_n, w) on n vertices. Proposition 3.1.3 will establish the relation between $\gamma(K_n, w)$ and $w_{\max}(K_n, w)$; in particular, we will work out a strategy for Alice for the cases $n-1 \leq w_{\max}(K_n, w) - w_{\max_2}(K_n, w)$ as well as $w_{\max}(K_n, w) - w_{\min}(K_n, w) \leq n-2$. Afterwards, we will turn our attention to a distribution of vertex-weights which arises from the mapping $w : V \to \{k, l\}$ such that we are interested in finding out $\gamma(K_n, w)$ in respect of the frequency k and l appear.

First, we start with a general estimation of $\gamma(K_n, w)$ and prove that the parameter is bounded between $w_{\max}(K_n, w)$ and $w_{\max}(K_n, w) + n - 1$, irrespective of the distribution of the vertex-weights. One may conjecture that a winning strategy for Alice has to be worked out. However, the structure of a complete graph, which implies that all vertices are connected with each other, renders any strategy of Alice unnecessary.

Proposition 3.1.1. *Let $(K_n, w) = (V, E, w)$ be a weighted complete graph. Then it holds*

$$w_{\max}(K_n, w) \leq \gamma(K_n, w) \leq w_{\max}(K_n, w) + n - 1.$$

Proof. The inequality $w_{\max}(K_n, w) \leq \gamma(K_n, w)$ is trivial, since a proper coloring is guaranteed if and only if for each vertex $v \in V(K_n, w)$ it holds $c(v) \geq w(v)$, where $c : V \to \mathbb{N}$ is the mapping which assigns to each vertex a feasible color. Thus, at least $w_{\max}(K_n, w)$ colors are required in order to color (K_n, w) no matter which strategies Alice and Bob apply.

The distribution of the vertex-weights which increases $\gamma(K_n, w)$ at most is $w : V \to \{w_{\max}(G, w)\}$, that is if to each vertex is assigned the same weight. Since (K_n, w) is complete, the vertices achieve distinct colors. Hence, $w_{\max}(K_n, w) + n - 1$ colors suffice in order to color the graph. □

Remark 3.1.2. The following tabular demonstrates a possible outcome of the game on the graph from proposition 3.1.1. Let $V = \{x_1, ..., x_n\}$.

Step	Vertex	Color
1	x_1	$w_{\max}(K_n, w)$
2	x_2	$w_{\max}(K_n, w) + 1$
3	x_3	$w_{\max}(K_n, w) + 2$
$\vdots$	$\vdots$	$\vdots$
i	x_i	$w_{\max}(K_n, w) + i - 1$
$\vdots$	$\vdots$	$\vdots$
n	x_n	$w_{\max}(K_n, w) + n - 1$

In the following we bound the number of the vertices of (K_n, w) in terms of parameters $\lambda := w_{\max}(K_n, w) - w_{\max_2}(K_n, w) > 0$ and $\mu := w_{\max}(K_n, w) - w_{\min}(K_n, w)$, where λ is the difference of the two greatest weights of (K_n, w) and μ the difference of the maximum and minimum weight.

Proposition 3.1.3. *Let $(K_n, w) = (V, E, w)$ be a weighted complete graph.*

(i) If $n - 1 \leq \lambda$ and $|\{v \in V \mid w(v) = w_{\max}(K_n, w)\}| = 1$, then $\gamma(K_n, w) = w_{\max}(K_n, w)$.

(ii) If $\mu \leq n - 2$, then $\gamma(K_n, w) > w_{\max}(K_n, w)$.

Proof. Let $V(K_n, w) = \{v_1, v_2, ..., v_n\}$.

(i) Suppose v_i is the vertex with $w(v_i) = w_{\max}(K_n, w)$ and assume that $w(v_i)$ colors are given.

Alice's strategy: Initially, Alice colors vertex v_i with $w(v_i)$. Throughout the game she colors the vertices by assigning them the smallest feasible color not taking the weight distribution into account.

If Alice applies the strategy illustrated above, then she wins the game because of the following consideration:

Consider the subgraph $(K_n, w) - \{v_i\} = (K_{n-1}, w)$ with $n - 1$ uncolored vertices, where v_i is colored. Since $w(v_i) - w_{\max_2}(K_n, w) \geq n - 1$, there are at least as many colors left as uncolored vertices. Thus, by the assumption $|\{v \in V \mid w(v) = w_{\max}(K_n, w)\}| = 1$ and because of the pigeon hole principle $w(v_i) - 1$ colors suffice to color $\{v_1, v_2, ..., v_{i-1}, v_{i+1}, ..., v_n\}$.

If Alice started the game by coloring a vertex $v_j \neq v_i$, then Bob could assign to a vertex $v_k \neq v_i$ the color $w(v_i)$. In that case since $(v_i, v_k) \in E$, there won't be a color left for v_i and a proper coloring would not be possible anymore.

(ii) Suppose that $w_{\max}(K_n, w)$ colors are given. Since all vertices of the graph are adjacent to each other, more than $w_{\max}(K_n, w)$ colors are required, if we admit $|\{v \in V \mid w(v) = w_{\max}(K_n, w)\}| > 1$. Thus, without loss of generality let $|\{v \in V \mid w(v) = w_{\max}(K_n, w)\}| = 1$ and assume that $w(v_i) = w_{\max}(K_n, w)$. Further, it is sufficient to assume that the vertices of (K_n, w) achieve only values from $\{w_{\max}(K_n, w), w_{\min}(K_n, w)\}$ since any other weight would lead to an increase of the required number of colors for a proper coloring. We will give a winning strategy for Bob such that a proper coloring of the graph is not possible, if $w_{\max}(K_n, w)$ colors are given.

Bob's strategy: By symmetry, in her first turn Alice colored either v_i or an arbitrary vertex from $\{v_1, ..., v_{i-1}, v_{i+1}, ..., v_n\}$.

- Assume that she colored $v_j \neq v_i$. Then, Bob colors an arbitrary vertex $v_k \neq v_i$ with the color $w(v_i)$ and avoids Alice's victory.
- Assume that Alice colored v_i with the color $w(v_i)$. Consider the subgraph $(K_{n-1}, w) = (K_n, w) - \{v_i\}$ where each vertex has the same weight $w_{\min}(K_n, w) = w_{\max}((K_n, w) - \{v_i\})$. Thus, by the proposition 3.1.1 we can conclude that

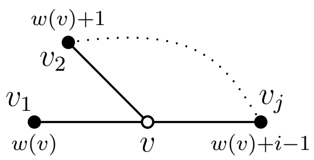
$$\begin{aligned}
\gamma((K_n, w) - \{v_i\}) &= w_{\max}((K_n, w) - \{v_i\}) + (n-1) - 1 \\
&= w_{\min}(K_n, w) + n - 2 \\
&\geq w_{\min}(K_n, w) + \mu \\
&= w_{\max}(K_n, w).
\end{aligned}$$

Thus, $w_{\max}(K_n, w)$ colors won't be sufficient to color (K_n, w). □

Further we consider distributions of vertex-weights on (K_n, w), where we restrict the mapping $w : V \to \mathbb{N}$ by two distinct vertex-weights k and l for $k > l$. Thus, we assume that (K_n, w) contains only a maximum weight $k = w_{\max}(K_n, w)$ and a minimum weight $l = w_{\min}(K_n, w)$. In particular, it is of our interest to figure out how Alice's strategy varies with the distribution of the weights.

From now on we will partition the vertex set V by $V = \{V^k, V^l\}$, where V^k is the set of vertices with vertex-weight k and V^l is the set of the vertices with vertex-weight l. Further let $p = |V^k|$ and $q = |V^l|$.

Definition 3.1.4. Let (G, w) be a weighted graph with $v \in V$ and $N(v)$ the set of all neighbors of v. Assume that v is not colored. We say that v is *attacked for i times*, if j vertices from $N(v)$ for $j \geq i$ are colored with i distinct colors of the form $\{w(v), w(v)+1, ..., w(v)+i-1\}$.



Proposition 3.1.5. *Let $(K_n, w) = (V, E, w)$ be a weighted complete graph with $w : V \to \{k, l\}$ where $k > l$ and let $p, q > 0$. Then the following holds:*

$$\gamma(K_n, w) = \{k + 2(p-1),\ \ n + l - 1,\ \ \left\lfloor \frac{q}{2} \right\rfloor + k + p - 1\}.$$

Proof. We will give Bob's optimal strategy as well as two strategies for Alice, σ_1 and σ_2, and prove why they turn out to be optimal. Afterwards, we will determine the required number of colors for coloring the graph, while assuming that Alice applies σ_1 and σ_2. In particular, the cases $p < q$ and $p \geq q$ are considered. Let C be the set of colored vertices.

Bob's strategy: By the structure of the graph, Bob's optimal strategy is obvious:

- If there exists a vertex $u \in V^l$ such that $u \notin C$, then color u with the greatest available color.

- If $V^l \in C$, then color vertices at random.

Let $v \in V^k$ be the last vertex with weight k that is being colored. Then by applying his strategy Bob achieves to attack v at most.

strategy σ_1:

- Let $V^k \not\subset C$. Then color $v \in V^k$ for $v \notin C$ with the smallest feasible color.
- Let $V^k \subset C$ and $V^l \not\subset C$. Then color $u \in V^l$ for $u \notin C$ with the smallest feasible color.

strategy σ_2:

- Let $V^l \not\subset C$. Then color $u \in V^l$ for $u \notin C$ with the smallest feasible color.
- Let $V^l \subset C$ and $V^k \not\subset C$. Then color $v \in V^k$ for $v \notin C$ with the smallest feasible color.

While strategy σ_1 implies that Alice colors first vertices from V^k until each vertex with weight k is colored, in strategy σ_2 she starts coloring V^k as soon as each vertex with weight l is colored.

Consider strategy σ_3 where Alice colors the vertices arbitrarily not taking their weights into account. Assume that σ_3 is an optimal strategy for Alice. Bob's worst case is obvious: He colors vertices with weight l with colors greater or equal to k. Once every vertex with weight l or every vertex with weight k is colored, the rest of the coloring game is trivial. Suppose Alice applies σ_3 and that at some point of the game every vertex with weight l is colored. If Alice applied σ_2, she would reach the configuration that each vertex with weight l is colored earlier. Thus, Bob achieves less attacks, if Alice applied σ_2 than σ_3. Hence, σ_3 cannot be optimal. The same conclusion can be drawn with the configuration that each vertex with weight k is colored and σ_1. Finally, this shows that only σ_1 or σ_2 can be optimal.

We are left with the task of determining the required number of colors such that Alice achieves a feasible coloring of the graph, if she plays σ_1 and σ_2. It is clear that in case $p \geq q$ the optimal strategy for Alice is σ_2. If she played σ_1, Bob could use every vertex with weight l to attack vertices with weight k. Thus, for the remainder of the proof let $p < q$.

Assume Alice applies strategy σ_1. Then by his strategy Bob makes $p-1$ attacks on V^k by assigning to $p-1$ vertices from V^l colors greater or equal to $k+1$. Since by the definition of a feasible coloring of a weighted graph each vertex has been colored with a color greater or equal to its weight, $k+p-1+p-1 = k+2(p-1)$ colors are required for coloring V^k. Then, $x = q-(p-1)$ uncolored vertices with weight l remain.

- Let $k - l \geq q - p + 1$.

  Then there are at least as many available colors left between k and l as uncolored vertices from V^l. Thus, it is clear that $k+2(p-1)$ colors suffice in order to color the graph.

- Let $k - l < q - p + 1$ and $y = k - l$.

  Since there are more uncolored vertices from V^l left as colors between k and l, the required number of the colors increases by $x - y$, such that

$$
\begin{aligned}
k + 2(p-1) + (x-y) &= k + 2(p-1) + q - (p-1) - (k-l) \\
&= k + 2p - 2 + q - p + 1 - k + l \\
&= n + l - 1.
\end{aligned}
$$

  colors are sufficient to color the graph.

Assume Alice applies strategy σ_2. Since Alice starts the game, Bob attacks V^k for $\lfloor \frac{q}{2} \rfloor$ times. Hence, $\lfloor \frac{q}{2} \rfloor + k + p - 1$ colors are needed for coloring V^k.

Now consider the coloring of V^l:

- Let $k-l \geq \lceil \frac{q}{2} \rceil$. Since Bob is using colors greater or equal to k for coloring V^l the colors between k and l suffice in order to color all vertices from V^l. Thus, $\lfloor \frac{q}{2} \rfloor + k + p - 1$ colors suffice for achieving a proper coloring of the entire graph.

- Let $k - l < \lceil \frac{q}{2} \rceil$. Then obviously the colors between k and l are not sufficient for coloring all vertices from V^l. Thus, $\lceil \frac{q}{2} \rceil - (k-l)$ more colors are required for achieving a proper coloring of the entire graph, such that

$$\left\lfloor \frac{q}{2} \right\rfloor + k + p - 1 + \left\lceil \frac{q}{2} \right\rceil - (k-l) = n + l - 1$$

colors are sufficient for Alice's victory.

Further, we shall make a case differentiation in $k-l$ by considering the cases $q-p+1 < \lceil \frac{q}{2} \rceil$, $q-p+1 > \lceil \frac{q}{2} \rceil$ and $q-p+1 = \lceil \frac{q}{2} \rceil$.

Let $q-p+1 < \lceil \frac{q}{2} \rceil$. See the following figure which demonstrates for all values of $k-l$ the required colors for Alice's victory if she applies strategy σ_1, respectively, σ_2:

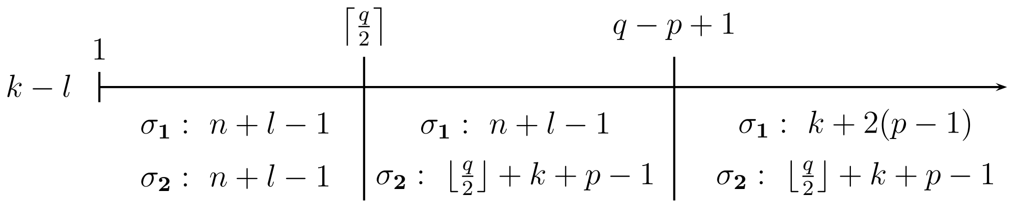
$k-l$: from 1	from $q-p+1$	from $\lceil \frac{q}{2} \rceil$
$\sigma_1:\ n+l-1$	$\sigma_1:\ k+2(p-1)$	$\sigma_1:\ k+2(p-1)$
$\sigma_2:\ n+l-1$	$\sigma_2:\ n+l-1$	$\sigma_2:\ \lfloor \frac{q}{2} \rfloor + k + p - 1$

- Let $k-l < q-p+1$. Then Alice is indifferent wether to play σ_1 or σ_2.

- Let $q-p+1 \leq k-l < \lceil \frac{q}{2} \rceil$. If $k+2(p-1) < n+l-1$, then Alice plays σ_1. If $k+2(p-1) > n+l-1$, she plays σ_2. In case $k+2(p-1) = n+l-1$, she is indifferent which strategy to apply.

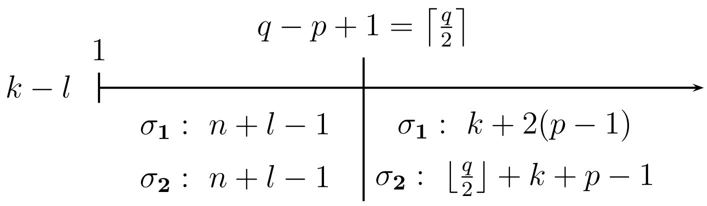
- Let $k-l \geq \lceil \frac{q}{2} \rceil$. If $k+2(p-1) < \lfloor \frac{q}{2} \rfloor + k + p - 1$, then Alice plays σ_1. If $k+2(p-1) > \lfloor \frac{q}{2} \rfloor + k + p - 1$, she plays σ_2. In case $k+2(p-1) = \lfloor \frac{q}{2} \rfloor + k + p - 1$, she is indifferent which strategy to apply.

Let $q-p+1 > \lceil \frac{q}{2} \rceil$. See the following figure:



- Let $k-l < \lceil \frac{q}{2} \rceil$. Then Alice is indifferent wether to play σ_1 or σ_2.
- Let $\lceil \frac{q}{2} \rceil \leq k-l < q-p+1$. If $n+l-1 < \lfloor \frac{q}{2} \rfloor + k + p - 1$, then Alice plays σ_1. If $n+l-1 > \lfloor \frac{q}{2} \rfloor + k + p - 1$, she plays σ_2. In case $n+l-1 = \lfloor \frac{q}{2} \rfloor + k + p - 1$, she is indifferent which strategy to apply.
- Let $k-l \geq q-p+1$. If $k+2(p-1) < \lfloor \frac{q}{2} \rfloor + k + p - 1$, then Alice plays σ_1. If $k+2(p-1) > \lfloor \frac{q}{2} \rfloor + k + p - 1$, she plays σ_2. In case $k+2(p-1) = \lfloor \frac{q}{2} \rfloor + k + p - 1$, she is indifferent which strategy to apply.

Let $q-p+1 = \lceil \frac{q}{2} \rceil$. See the following figure:



- If $k-l < \lceil \frac{q}{2} \rceil$, Alice is indifferent wether she plays σ_1 or σ_2.
- Let $k-l \geq \lceil \frac{q}{2} \rceil$. If $k+2(p-1) < \lfloor \frac{q}{2} \rfloor + k + p - 1$, then Alice plays σ_1. If $k+2(p-1) > \lfloor \frac{q}{2} \rfloor + k + p - 1$, she plays σ_2. In case $k+2(p-1) = \lfloor \frac{q}{2} \rfloor + k + p - 1$, she is indifferent which strategy to apply.

$\square$

3.2 The Game Chromatic Number of Weighted Complete Multipartite Graphs

In this section we will discuss the two-person game on weighted complete multipartite graphs relating to certain distributions of the vertex-weights. We will consider the graph $(K_{s_1,...,s_n}, w)$ where $S_1, ..., S_n$ are the independent sets with $|S_i| = s_i$ for $i \in \{1, ..., n\}$. For simplicity of notation we will write $w(S_i) = k$ for $i \in \{1, ..., n\}$, if $w(v_{i_j}) = k$ for all $j \in \{1, ..., s_i\}$. Further, let $w_{\max}(S_i)$ be the greatest weight which is assigned to a vertex in S_i. Since for $u, v \in S_i$ it holds $(u, v) \notin E$, the coloring of the set S_i is fixed as soon to an arbitrary vertex is assigned a color greater or equal to $w_{\max}(S_i)$.

We will introduce the so called *two-weighted complete multipartite graphs*, where $w(S_i) = \{k, l\}$ with $|\{S_i \mid w(S_i) = k\}| \geq 1$ and $|\{S_i \mid w(S_i) = l\}| \geq 1$, and analyze the game chromatic number. It is worth pointing out that Alice's winning strategy changes if we consider the two cases $s_i \geq 4$ and $s_i \geq 3$ for all $i \in \{1, ..., n\}$. We will work out winning strategies for Alice for the cases $s_i \geq 4$ and $s_i \geq 3$ in the propositions 3.2.4 and 3.2.5, respectively. Afterwards, we will generalize the notion of the two-weighted complete multipartite graphs and determine the game chromatic number of *n-weighted complete multipartite graphs*, where $w(S_i) \neq w(S_j)$ for all $i, j \in \{1, ..., n\}$ holds. Finally, we will introduce the notion of *(k, l)-weighted graphs*, where the vertices of an independent set are assigned to distinct weights k and l.

While Alice's strategy varies by the structure of the graph, Bob's strategy stays the same. The following strategy demonstrates the worst case scenario.

strategy σ: Let $v \in S_i$ for $i \in \{1, ..., n\}$ be the vertex that Alice colored in her last turn. Then Bob colors $u \in S_i$ with the greatest available color different than the color of v.

In particular, σ increases the required number of colors at most because of the following consideration.

- If Bob colored in an independent set S_h for $h \neq i$ which contains only uncolored vertices, he would waste an attack, since the set S_h would be "saved" from Alice's view.

- If he colored in an independent set that contains colored vertices with an already used color, then the result of the game would be the chromatic number and that is clearly not optimal for Bob.

First, let us introduce a general estimation:

Proposition 3.2.1. *Let $(K_{s_1,...,s_n}, w) = (V, E, w)$ be a weighted complete multipartite graph with $s_i \geq 4$ for all $i \in \{1, ..., n\}$. Then*

$$w_{\max}(K_{s_1,...,s_n}, w) \leq \gamma(K_{s_1,...,s_n}, w) \leq w_{\max}(K_{s_1,...,s_n}, w) + 2n - 2.$$

Proof. It is easy to see that the inequality $w_{\max}(K_{s_1,...,s_n}, w) \leq \gamma(K_{s_1,...,s_n}, w)$ holds, since one needs at least $w_{\max}(K_{s_1,...,s_n}, w)$ colors in order to obtain a feasible coloring $c : V \to \{1, ..., k\}$ with $c(v) \geq w(v)$ for all $v \in V$.

In order to show the inequality $\gamma(K_{s_1,...,s_n}, w) \leq w_{\max}(K_{s_1,...,s_n}, w) + 2n - 2$ we consider the case $w(v) = k$ for all $v \in V$ because obviously this distribution maximizes the required number of colors for coloring the graph.

Because of the pigeon hole principle obviously at least $k - 1 + n$ colors are required, since $(K_{s_1,...,s_n}, w)$ is complete. Since each vertex of the graph has the same weight k, we can refer to Alice's winning strategy worked out in 2.3.2 for the choice of the vertices to color. However, in our case one has to color the vertices with integers greater or equal to k with first-fit and not assign them to unit length arcs on C^r. Thus, each time Alice takes turn she colors a vertex from an independent set which consists only of uncolored vertices, while Bob applies strategy σ. As soon as each independent set contains at least one colored vertex the coloring of the remaining uncolored vertices is trivial. This implies that Bob will attack for $n - 1$ times. Hence, we can conclude the following calculation.

$$\begin{aligned}
\gamma(K_{s_1,...,s_n}, w) &\leq k - 1 + n + (n - 1) \\
&= k + 2n - 2 \\
&= w_{\max}(K_{s_1,...,s_n}, w) + 2n - 2.
\end{aligned}$$

□

Corollary 3.2.2. *Let* $(K_{s_1,s_2}, w) = (V, E, w)$ *be a weighted complete bipartite graph. Then*

$$w_{\max}(K_{s_1,s_2}, w) \leq \gamma(K_{s_1,s_2}, w) \leq w_{\max}(K_{s_1,s_2}, w) + 2. \qquad \square$$

Definition 3.2.3. We call $(K_{s_1,\ldots,s_n}, w) = (V, E, w)$ a *two-weighted complete multipartite graph* when the following holds:

(i) The weight function is of the form $w : V \to \{k, l\}$ with $k > l$.

(ii) Vertices belonging to the same independent set assume the same vertex-weight.

For a two-weighted complete multipartite graph $(K_{s_1,\ldots,s_n}, w)$ we classify the number of the independent sets into $n = p+q$, where $p = |\{S_i \mid w(S_i) = k\}|$ and $q = |\{S_i \mid w(S_i) = l\}|$ for $i \in \{1, \ldots, n\}$.

Proposition 3.2.4. *Let* $(K_{s_1,\ldots,s_n}, w)$ *be a two-weighted complete multipartite graph with* $s_i \geq 4$ *for all* $i \in \{1, \ldots, n\}$.

(i) If $\sum_{\{v \in V \mid w(v)=k\}} 1 = 2m$ *for* $m \in \mathbb{N}$, *then*

$$\gamma(K_{s_1,\ldots,s_n}, w) = \begin{cases} k + 2p - 2, & \textit{for} \;\; k - l \geq 2q - 1, \\ 2n + l - 2, & \textit{for} \;\; k - l < 2q - 1. \end{cases}$$

(ii) If $\sum_{\{v \in V \mid w(v)=k\}} 1 = 2m + 1$ *for* $m \in \mathbb{N}$, *then*

$$\gamma(K_{s_1,\ldots,s_n}, w) = \begin{cases} k + 2p - 2, & \textit{for} \;\; k - l \geq 2q - 2, \\ 2n + l - 2, & \textit{for} \;\; k - l < 2q - 2. \end{cases}$$

Proof. We will prove a winning strategy for Alice such that she wins the game with the given number of colors. Without loss of generality assume that $w(S_i) = k$ for all $i \in \{1, \ldots, p\}$ and $w(S_j) = l$ for all $j \in \{p+1, \ldots, n\}$. Let C be the set of all colored vertices.

(i) Assume that $\sum_{\{v \in V \mid w(v)=k\}} 1 = 2m$ for $m \in \mathbb{N}$.

Alice's strategy: Initially, Alice colors vertex $v \in S_1$ with the smallest feasible color which is k. Then she proceeds as follows:

- If there exists a $j \in \{2, ..., p\}$ with $S_j \cap C = \emptyset$, then she colors $u \in S_j$ for $u \notin C$ with the smallest feasible color.

- If $S_j \cap C \neq \emptyset$ for all $j \in \{2, ..., p\}$ and if there exists a j_0 with $S_{j_0} \cap C = \emptyset$ for $j_0 \in \{p+1, ..., n\}$, then Alice colors $u \in S_{j_0}$ for $u \notin C$ with the smallest feasible color.

- Otherwise, if $S_j \cap C \neq \emptyset$ for all $j \in \{1, ..., n\}$, the coloring is fixed.

Alice's winning strategy implies that she aims to start coloring vertices with weight l as soon as each independent set from $\{S_1, ..., S_p\}$ contains at least one colored vertex. If she colored a vertex with vertex-weight l and there existed S_{i_0} for $i_0 \in \{1, ..., p\}$ with $S_{i_0} \cap C = \emptyset$, then due to Bob's worst case strategy σ he would color another vertex with weight l from the same independent set with colors greater or equal to k. Obviously, this increases the number of the required colors by 1 because S_{i_0} would be attacked once.

The strategy in which Alice colors in an independent set that contains colored vertices is not considered. Obviously, this case is not optimal because Bob could maximize the amount of his attacks. Thus, the strategy illustrated above turn, out to be optimal.

We can draw the following conclusions. Since the S_i's for $i \in \{1, ..., p\}$ are colored first, $(k-1) + (2p-1) = k + 2p - 2$ colors suffice in order to color all S_i for $i \in \{1, ..., p\}$, if we assume that the last S_i which is colored received only one color. For the case that there are more than $k + 2p - 2$ colors given, Bob would also color the last S_i for $i \in \{1, ..., p\}$ with a second color. Thus, $k + 2p - 1$ colors would be required for all S_i's with $i \in \{1, ..., p\}$. Further, for coloring $\{S_{p+1}, ..., S_n\}$, $2q - 1$ colors suffice by the optimal strategies of the players since every S_j for $j \in \{p + 1, ..., n\}$ receives two colors, except the last one receiving one color.

- Let $k-l \geq 2q-1$ and assume that $k+2p-2$ colors are given. Then there are enough colors between l and $k-1$ to color all the S_j's for $j \in \{p+1, ..., n\}$.

Thus,

$$\gamma(K_{s_1,\dots,s_n}, w) = k + 2p - 2.$$

- Let $k - l < 2q - 1$ and assume that $2n + l - 2$ colors are given. Then there are not enough colors between l and $k - 1$ to color all the S_j's for $j \in \{p + 1, \dots, n\}$ because one needs at least $2q - 1$ colors for coloring $\{S_{p+1}, \dots, S_n\}$. Therefore, $(2q - 1) - (k - l)$ additional colors are required. Hence,

$$\begin{aligned}\gamma(K_{s_1,\dots,s_n}, w) &= k + 2p - 1 + (2q - 1) - (k - l)\\ &= 2p + 2q + l - 2\\ &= 2n + l - 2.\end{aligned}$$

(ii) Let $\sum_{\{v \in V \mid w(v)=k\}} 1 = 2m + 1$ for $m \in \mathbb{N}$.

- Let $k - l \geq 2q - 2$ and assume that $k + 2p - 2$ colors are given. Then Alice's optimal strategy is similar as above with one decisive difference: As soon as each independent set from $\{S_1, \dots, S_p\}$ contains a colored vertex, Alice does not proceed with coloring $\{S_{p+1}, \dots, S_n\}$ but colors the remaining uncolored vertices with weight k until $\{S_1, \dots, S_p\} \subset C$. Thus, by strategy σ Bob is forced to color a vertex with weight l first, by the assumption $\sum_{\{v \in V \mid w(v)=k\}} 1 = 2m+1$. This implies that he loses the possibility to attack the l-weighted vertices once. Afterwards, Alice jumps to an uncolored S_j for $j \in \{p + 1, \dots, n\}$. Thus, the run of the game is the same as above. Because Bob was forced to jump to an S_j for $j \in \{p+1, \dots, n\}$, the difference $k - l$ suffices to be $2q - 2$ instead of $2q - 1$ as in the case before.

- Let $k - l < 2q - 2$ and assume than $2n + l - 2$ colors are given. Obviously, it does not make sense for Alice to start coloring $\{S_{p+1}, \dots, S_n\}$ as soon as each k-weighted vertex is colored because more than $k + 2p - 2$ colors are given. Otherwise, Bob would attack uncolored independent sets from $\{S_{p+1}, \dots, S_n\}$. Thus, we can apply the same optimal strategies as in (i) and hence the proof runs as above. $\square$

The following proposition demonstrates that even a slight change of the structure of the graph leads to a modification of Alice's winning strategy. So far, we have been working under the assumption that $s_i \geq 4$ for all $i \in \{1, ..., n\}$. In the following we will admit that there exist independent sets with at least three vertices. Let us define $\tilde{p} = |\{S_i \mid w(S_i) = k \text{ and } s_i = 3\}|$ and $\tilde{q} = |\{S_j \mid w(S_j) = l \text{ and } s_j = 3\}|$. Note that p and q are the numbers of independent sets with weight k and l. In particular, we will introduce winning strategies for Alice for the coloring game considering the cases $\tilde{p} \geq 1 \wedge \tilde{q} \geq 1$, $\tilde{p} \geq 1 \wedge \tilde{q} = 0$ and $\tilde{p} = 0 \wedge \tilde{q} \geq 1$.

Proposition 3.2.5. *Let $(K_{s_1,...,s_n}, w)$ be a two-weighted complete multipartite graph with $s_i \geq 3$ for $i \in \{1, ..., n\}$.*

(i) For $\sum_{\{v \in V \mid w(v)=k\}} 1 = 2m$, $m \in \mathbb{N}$, the following cases hold:

Case 1.a: Let $\tilde{p} \geq 1$ and $\tilde{q} \geq 1$, then

$$\gamma(K_{s_1,...,s_n}, w) = \begin{cases} k + 2p - 2, & if \;\; k - l \geq 2q - 2, \\ 2n + l - 3, & if \;\; k - l < 2q - 2. \end{cases}$$

Case 1.b: Let $\tilde{p} \geq 1$ and $\tilde{q} = 0$, then

$$\gamma(K_{s_1,...,s_n}, w) = \begin{cases} k + 2p - 2, & if \;\; k - l \geq 2q - 1, \\ 2n + l - 3, & if \;\; k - l < 2q - 1. \end{cases}$$

Case 2.a: Let $\tilde{p} = 0$ and $\tilde{q} \geq 1$, then

$$\gamma(K_{s_1,...,s_n}, w) = \begin{cases} k + 2p - 2, & if \;\; k - l \geq 2q - 2, \\ 2n + l - 2, & if \;\; k - l < 2q - 2. \end{cases}$$

Case 2.b: Let $\tilde{p} = 0$ and $\tilde{q} = 0$, then

$$\gamma(K_{s_1,...,s_n}, w) = \begin{cases} k + 2p - 2, & if \;\; k - l \geq 2q - 1, \\ 2n + l - 2, & if \;\; k - l < 2q - 1. \end{cases}$$

(ii) For $\sum_{\{v \in V \mid w(v)=k\}} 1 = 2m + 1$, $m \in \mathbb{N}$, the following cases hold:

<u>Case 1.a:</u> Let $\tilde{p} \geq 1$ and $\tilde{q} \geq 1$, then

$$\gamma(K_{s_1,...,s_n}, w) = \begin{cases} k + 2p - 2, & if \;\; k - l \geq 2q - 2, \\ 2n + l - 3, & if \;\; k - l < 2q - 2. \end{cases}$$

<u>Case 1.b:</u> Let $\tilde{p} \geq 1$ and $\tilde{q} = 0$, then

$$\gamma(K_{s_1,...,s_n}, w) = \begin{cases} k + 2p - 2, & if \;\; k - l \geq 2q - 2, \\ 2n + l - 3, & if \;\; k - l < 2q - 2. \end{cases}$$

<u>Case 2.a:</u> Let $\tilde{p} = 0$ and $\tilde{q} \geq 1$, then

$$\gamma(K_{s_1,...,s_n}, w) = \begin{cases} k + 2p - 2, & if \;\; k - l \geq 2q - 2, \\ 2n + l - 2, & if \;\; k - l < 2q - 2. \end{cases}$$

<u>Case 2.b:</u> Let $\tilde{p} = 0$ and $\tilde{q} = 0$, then

$$\gamma(K_{s_1,...,s_n}, w) = \begin{cases} k + 2p - 2, & if \;\; k - l \geq 2q - 2, \\ 2n + l - 2, & if \;\; k - l < 2q - 2. \end{cases}$$

Proof. Without loss of generality let $(K_{s_1,...,s_n}, w)$ be a two-weighted complete multipartite graph with $w(S_i) = k$ for all $i \in \{1, ..., p\}$ and $w(S_j) = l$ for all $j \in \{p+1, ..., n\}$ for $k, l \in \mathbb{N}$ and $k > l$. We will prove a winning strategy for Alice for all cases except 2.b in (i) and (ii) since the case $\tilde{p} = \tilde{q} = 0$ results in the graph that we considered in 3.2.4. Thus, we will drop the proofs. For Bob's optimal strategy, we refer to σ on page 45. Further, let C be the set of all colored vertices which is obviously empty at the beginning of the game.

(i) Let $\sum\limits_{\{v \in V \mid w(v) = k\}} 1 = 2m$ for $m \in \mathbb{N}$.

<u>Case 1.a</u>: Let $\tilde{p} \geq 1$ and $\tilde{q} \geq 1$. Without loss of generality suppose that $s_1 = s_{p+1} = 3$. In particular, let $S_1 = \{v_1, v_2, v_3\}$ and $S_{p+1} = \{u_1, u_2, u_3\}$. Let $k - l \geq 2q - 2$ and assume that $k + 2p - 2$ colors are given. We will denote by σ_k^1 and σ_l^1 Alice's strategies of coloring $\{S_1, ..., S_p\}$ and $\{S_{p+1}, ..., S_n\}$, respectively.

σ_k^1: Initially, Alice colors a vertex from $\{S_1, ..., S_p\}$ with the smallest available color, which is k. Then she proceeds as follows:

- Assume that there exists an $i_0 \in \{1, ..., p\}$ such that $S_{i_0} \cap C = \emptyset$. Then she colors $v \in S_{i_0}$ with the smallest feasible color.
- Assume that $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$. If there exists a $j_0 \in \{1, ..., p\}$ such that $S_{j_0} \not\subset C$, then she colors $v' \in S_{j_0}$ with the smallest feasible color. If $S_i \subset C$ for all $i \in \{1, ...p\}$, then she proceeds coloring vertices from $\{S_{p+1}, ..., S_n\}$.

Bob's worst case strategy σ implies the following run of the game: Once $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$, Alice and Bob assigned p and $p-1$ distinct colors, respectively. Thus, $k-1+p+(p-1) = k+2p-2$ colors suffice for achieving a proper coloring of $\{S_1, ..., S_p\}$. If there exists an i_0 such that $S_{i_0} \cap C = \emptyset$ and Alice colored a vertex $x \in S_{j_0}$,where $S_{j_0} \cap C \neq \emptyset$, then by strategy σ Bob would color a vertex $y \in S_{j_0}$ with $x \neq y$ with the greatest available color. This would increase the number of the required colors by 1 since $(y, z) \in E(K_{s_1,...,s_n}, w)$ for $z \in S_{i_0}$. However, they keep on coloring in $\{S_1, ..., S_p\}$ until $S_i \subset C$ for all $i \in \{1, ..., p\}$. By the assumption $\sum\limits_{\{v \in V \,|\, w(v)=k\}} 1 = 2m$ for $m \in \mathbb{N}$, Alice is the one who colors a vertex from $\{S_{p+1}, ..., S_n\}$ first.

σ_l^1: Initially, Alice colors vertex u_1 with the smallest feasible color followed by Bob who colors u_2 or u_3 by σ with $\{u_1, u_2, u_3\} \in S_{p+1}$. Without loss of generality we may assume that he colors u_2. Then Alice colors u_3 with the same color as u_1 or u_2. Then she proceeds as follows:

- Assume there exists an $i_0 \in \{p+2, ..., n\}$ such that $S_{i_0} \cap C = \emptyset$. Then she colors $v \in S_{i_0}$ with the smallest feasible color.
- Assume that $S_i \cap C \neq \emptyset$ for all $i \in \{p+2, ..., n\}$. Then she colors arbitrary vertices with the smallest feasible colors until each vertex of the graph is colored.

Bob's worst case strategy σ implies the following run of the game: Suppose that the players color along the ordering $\{S_{p+1}, .., S_n\}$. Since $s_{p+1} = 3$ and $S_i \subset C$ for all $i \in \{S_1, ..., S_p\}$, Bob is forced to jump to S_{p+2} first. Then due to σ_l^1 Alice proceeds with S_{p+3}. From now on the run of the game implies that each set from $\{S_{p+1}, ..., S_n\}$ is assigned to 2 distinct colors, except S_{p+2} and S_n. Thus, Alice and Bob assigned $q-1$ distinct colors to $\{S_{p+1}, ..., S_n\}$, respectively, until $S_i \cap C \neq \emptyset$ for all $i \in \{p+1, ..., n\}$. Then, according to the restriction $k - l \geq 2q - 2$, $k + 2p - 2$ colors suffice to color the graph $(K_{s_1,..,s_n}, w)$.

Let $k - l < 2q - 2$ and assume that $2n + l - 3$ colors are given. We will denote by σ_k^2 and σ_l^2 Alice's strategies of coloring $\{S_1, ..., S_p\}$ and $\{S_{p+1}, ..., S_n\}$, respectively.

σ_k^2: Initially, Alice colors vertex $v_1 \in S_1$ with the smallest available color, which is k. Due to strategy σ Bob colors an arbitrary vertex from S_1 with a color different than k. Without loss of generality assume that he colors vertex v_2 with $k+1$. Then Alice colors v_3 with k or $k+1$ and proceeds as follows:

- Assume that there exists an $i_0 \in \{2, ..., p\}$ such that $S_{i_0} \cap C = \emptyset$. Then she colors $v \in S_{i_0}$ with the smallest feasible color.
- Assume that $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$. Then she proceeds coloring vertices from $\{S_{p+1}, ..., S_n\}$.

Bob's worst case strategy σ implies the following run of the game: By the assumption $s_1 = 3$, once $S_i \cap C \neq \emptyset$ for all $\in \{1, ..., p\}$ both assigned $p-1$ distinct colors to $\{S_1, ..., S_p\}$, respectively. But since more than $k - 1 + 2(p-1) = k + 2p - 3$ colors are given, Bob is able to assign a new color to a vertex with weight k. Thus, $k + 2p - 2$ colors are required for coloring $\{S_1, ..., S_p\}$. If Alice did not color v_3 but proceeded with coloring $\{S_2, ..., S_p\}$, then both would assign p colors to $\{S_1, ..., S_p\}$ such that $k + 2p - 1$ colors would be required for coloring $\{S_1, ..., S_p\}$. Further, if Alice kept on coloring vertices with weight k until $S_i \subset C$ for all $i \in \{1, ..., p\}$, then by the assumption $k - l < 2q - 2$ Bob would be able to attack the remaining uncolored vertices with weight l since more than $k + 2p - 2$ colors are given. Thus, a proper coloring of $\{S_{p+1}, ..., S_n\}$ won't be possible anymore .

σ_l^2: Initially, Alice assigns to an arbitrary vertex from $\{S_{p+1}, ..., S_n\}$ the smallest feasible color. Then she proceeds as follows:

- Assume there exists an $i_0 \in \{p+1, ..., n\}$ such that $S_{i_0} \cap C = \emptyset$. Then she colors $v \in S_{i_0}$ with the smallest feasible color.

- Assume that $S_i \cap C \neq \emptyset$ for all $i \in \{p+1, ..., n\}$. Then she colors arbitrary vertices with the smallest feasible colors until each vertex of the graph is colored.

Bob's worst case strategy σ implies the following run of the game: Once $S_i \cap C \neq \emptyset$ for all $i \in \{p+1, ..., n\}$, Alice and Bob assigned q and $q-1$ distinct colors to $\{S_{p+1}, ..., S_n\}$, respectively. Then, due to the restriction $k - l < 2q - 2,\ k + 2p - 2 + \big((2q - 1 - (k - l)\big) = 2n + l - 3$ colors are required for achieving a proper coloring of the graph. If there exists an $i_0 \in \{p+1, ..., n\}$ such that $S_{i_0} \cap C = \emptyset$ and Alice colored a vertex $x \in S_{j_0}$,where $S_{j_0} \cap C \neq \emptyset$, then by strategy σ Bob would color a vertex $y \in S_{j_0}$ with $x \neq y$ with the greatest available color. This would increase the number of the required colors by 1 since $(y, z) \in E(K_{s_1,...,s_n}, w)$ for $z \in S_{i_0}$.

Case 1.b: Let $\tilde{p} \geq 1$ and $\tilde{q} = 0$. Without loss of generality assume that $s_1 = 3$ and let $S_1 = \{v_1, v_2, v_3\}$. Let $k - l \geq 2q - 1$ and suppose that $k + 2p - 2$ colors are given.

Alice's strategy of coloring $\{S_1, ..., S_n\}$*:*

- Assume that there exists an $i_0 \in \{1, ..., p\}$ such that $S_{i_0} \cap C = \emptyset$. Then she colors $v \in S_{i_0}$ with the smallest feasible color.

- Assume that $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$. Then she goes on coloring $\{S_{p+1}, ..., S_n\}$.

- Assume there exists an $i_0 \in \{p+1, ..., n\}$ such that $S_{i_0} \cap C = \emptyset$. Then she colors $v \in S_{i_0}$ with the smallest feasible color.

- Assume that $S_i \cap C \neq \emptyset$ for all $i \in \{p+2, ..., n\}$. Then she colors arbitrary vertices with the smallest feasible colors until each vertex of the graph is colored.

By Bob's strategy σ Alice and Bob assign p and $p-1$ distinct colors to $\{S_1, ..., S_p\}$ until $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$. Thus, $k+2p-2$ colors suffice for achieving a proper coloring of $\{S_1, ..., S_p\}$. In contrast to the strategy σ_k^1 Alice does not have to color the remaining uncolored vertices with weight k because $s_i > 3$ for all $i \in \{p+1, ..., n\}$. Thus, again $k+2p-2$ colors suffice for coloring $\{S_1, ..., S_p\}$.

For the coloring of $\{S_{p+1}, ..., S_n\}$ we can conclude the following: Since $s_i > 3$ for all $i \in \{p+1, ..., n\}$, by Bob's strategy σ the run of the game is trivial. Alice and Bob assign q and $q-1$ distinct colors to $\{S_{p+1}, ..., S_n\}$, respectively. Thus, under the assumption $k-l \geq 2q-1,\ k+2p-2$ colors suffice for achieving a proper coloring for the entire graph.

Let $k-l < 2q-1$ and assume that $2n+l-3$ colors are given. Since $\tilde{p} \geq 1$, Alice's strategies for coloring $\{S_1, ..., S_p\}$ and $\{S_{p+1}, ..., S_n\}$ run by the same method as σ_k^2 and σ_l^2, respectively. Thus, $2p-1$ distinct colors are assigned to $\{S_1, ..., S_p\}$ and $2q-1$ distinct colors are assigned to $\{S_{p+1}, ..., S_n\}$ until $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., n\}$. Thus, $k+2p-2+(2q-1-(k-l)) = 2n+l-3$ suffice for achieving a proper coloring of the entire graph.

Case 2.a: Let $\tilde{p} = 0$ and $\tilde{q} \geq 1$. Let $k-l \geq 2q-2$ and assume that $k-l < 2q-2$ colors are given. Then by the same arguments as in case 1.a Alice applies strategies σ_k^1 and σ_l^1 for coloring $\{S_1, ..., S_p\}$ as well as $\{S_{p+1}, ..., S_n\}$, respectively.

Let $k-l < 2q-2$ and assume that $2n+l-2$ colors are given.

Alice's strategy of coloring $\{S_1, ..., S_p\}$*:*

- Assume that there exists an $i_0 \in \{1, ..., p\}$ such that $S_{i_0} \cap C = \emptyset$. Then she colors $v \in S_{i_0}$ with the smallest feasible color.
- Assume that $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., p\}$. Then she proceeds coloring vertices from $\{S_{p+1}, ..., S_n\}$.

Bob's strategy σ implies the following run of the game: In contrast to the strategy σ_k^2 it does not make sense for Alice to color twice in the same an independent set S_{i_0} for $i_0 \in \{1, ..., p\}$, while assuming that there exists a $j_0 \in \{1, ..., p\}$ such that $S_{j_0} \cap C = \emptyset$. Then by the assumption $s_i > 3$ for all $i \in \{1, ..., p\}$, Bob would assign to an arbitrary vertex from S_{i_0} a new color. Since vertices from S_{i_0} and S_{j_0} are adjacent, the number of the required colors for coloring $\{S_1, ..., S_p\}$ would increase by 1. Thus, both assign p distinct colors to $\{S_1, ..., S_p\}$, respectively, which implies that $k - 1 + 2p$ colors suffice.

Alice's strategy of coloring $\{S_{p+1}, ..., S_n\}$*:* Clearly, we can refer to strategy σ_l^2. Since the same conclusion can be drawn, we claim that $2q - 1$ times have to be colored until each set from $\{S_{p+1}, ..., S_n\}$ contains at least one colored vertex. Then, by the restriction $k - l < 2q - 2,\ \ k + 2p - 1 + (2q - 1 - (k - l)) = 2n + l - 2$ colors suffice for coloring $(K_{s_1,...,s_n}, w)$.

(ii) Let $\sum\limits_{\{v \in V \ |\ w(v)=k\}} 1 = 2m + 1$ for $m \in \mathbb{N}$.

Case 1.a: Let $\tilde{p} \geq 1$ and $\tilde{q} \geq 1$. Further, let $k - l \geq 2q - 2$ and assume that $k + 2p - 2$ colors are given. For coloring $\{S_1, ..., S_{p+1}\}$, Alice clearly applies strategy σ_k^1. Thus, Alice and Bob assign p and $p - 1$ distinct colors to $\{S_1, ..., S_p\}$ such that $k + 2p - 2$ colors suffice.

For the purpose of coloring $\{S_{p+1}, ..., S_n\}$, Alice clearly applies strategy σ_l^2. While assuming that Bob applies strategy σ, the run of the game is almost the same as in case 1.a except one decisive point. By the assumption $\sum\limits_{\{v \in V \ |\ w(v)=k\}} 1 = 2m + 1$ for $m \in \mathbb{N}$, the number of the vertices with weight k is odd. Thus, since Alice goes on coloring in $\{S_{p+1}, ..., S_n\}$ not before each vertex from $\{S_1, ..., S_p\}$ is colored, Bob is the one who has to color a vertex with weight l first. Without loss of generality assume that he colors a vertex in S_{p+1} which fixes the coloring of S_{p+1}. The further development of the game is trivial: each time Alice takes turn, she colors vertices from independent sets that contain only uncolored vertices with the smallest available colors followed by Bob who assigns a new color. Then, as soon as $S_i \cap C \neq \emptyset$, for all $i \in \{p+1, ..., n\}$, Alice and Bob assigned $q - 1$ distinct colors to $\{S_{p+1}, ..., S_n\}$, respectively. This implies that according to the restriction $k - l \geq 2q - 2,\ \ k + 2p - 2$ colors suffice to color the entire graph $(K_{s_1,..,s_n}, w)$.

Let $k - l < 2q - 2$ and assume that $2n + l - 3$. Then it is obvious that Alice's strategies for coloring $\{S_1, ..., S_p\}$ and $\{S_{p+1}, ..., S_n\}$ run by the same method as strategies σ_k^2 and σ_l^2, respectively.

<u>Case 1.b</u>: Let $\tilde{p} \geq 1$ and $\tilde{q} = 0$. Then, we can refer to (ii) case 1.a. The details are left to the reader.

<u>Case 2.a</u>: Let $\tilde{p} = 0$ and $\tilde{q} \geq 1$. Let $k - l \geq 2q - 2$ and assume that $k + 2p - 2$ colors are given. It is clear, that one can refer to (ii) case 1.a.

Let $k - l < 2q - 2$ and assume that $2n + l - 2$ colors are given. Then we can refer to (i) case 2.a. The details are left to the reader. □

In the previous proposition we dealt with the two-weighted complete multipartite graphs. Now, we go a step further and consider the general case. Thus, we restrict our attention to the *n-weighted complete multipartite graphs*. We define them as follows:

Definition 3.2.6. We call $(K_{s_1,...,s_n}, w) = (V, E, w)$ an *n-weighted complete multipartite graph* when the following holds:

(i) For each two vertices $u, v \in S_i$ for $i \in \{1, ..., n\}$, it holds $w(u) = w(v)$.

(ii) For each two vertices $u \in S_i$ and $v \in S_j$ for $i \neq j$ and $i, j \in \{1, ..., n\}$, it holds $w(u) \neq w(v)$.

In particular, we will consider the distributions $w(S_j) - w(S_{j-1}) = 1$ and $w(S_j) - w(S_{j-1}) \geq 2$ for all $j \in \{2, ...n\}$.

Proposition 3.2.7. *Let $(K_{s_1,...,s_n}, w) = (V, E, w)$ be an n-weighted complete multipartite graph with $s_i \geq 2$ for all $i \in \{1, ..., n\}$. Then the following hold:*

(i) If $w(S_j) - w(S_{j-1}) = 1$ for all $j \in \{2, ..., n\}$, then

$$\gamma(K_{s_1,...,s_n}, w) = w_{\max}(K_{s_1,...,s_n}, w) + n - 2.$$

(ii) If $w(S_j) - w(S_{j-1}) \geq 2$ for all $j \in \{2, ..., n\}$, then

$$\gamma(K_{s_1,...,s_n}, w) = w_{\max}(K_{s_1,...,s_n}, w).$$

Proof. (i) Let $w(S_1) = l$. Then, it holds $w(S_n) = w_{\max}(K_{s_1,...,s_n}, w) = l + n - 1$ by the assumption. Suppose that $w_{\max}(K_{s_1,...,s_n}, w) + n - 2 = l + 2n - 3$ colors are given. We will prove a winning strategy for Alice. In particular, we will refer to page 45 for Bob's strategy σ.

Alice's strategy: Let C be the set of colored vertices and assume that $\Pi(S_i)_{i\in\{1,...,n\}}$ is the set of all linear orderings on the independent sets of $\{S_1, ..., S_n\}$. Alice fixes the linear ordering $L = \{S_n, ..., S_1\}$ for $L \in \Pi(S_i)_{i\in\{1,...,n\}}$. Initially, she colors an arbitrary vertex $v \in S_n$ with the smallest feasible color which is $l+n-1$. Throughout the game she proceeds as follows.

- Assume that there exists at least one $i \in \{1, ..., n-1\}$ such that $S_i \cap C = \emptyset$. Then she searches the greatest $i_0 \in \{1, ..., n-1\}$ in L such that $S_{i_0} \cap C = \emptyset$ and colors an arbitrary vertex $v \in S_{i_0}$ with the smallest feasible color.
- Assume that $S_i \cap C \neq \emptyset$ for all $i \in \{1, ..., n\}$. Then she colors vertices at random.

Roughly speaking, Alice's strategy is to start coloring a vertex in S_n with the smallest feasible color and then to jump "backwards" to the next highest-weighted uncolored independent set in L. Assume that there exists an $i \in \{1, ..., n\}$ such that $S_i \cap C = \emptyset$. If Alice colored a vertex from an independent set S_j for $j < i$ or $S_j \cap C \neq \emptyset$, then Bob could color a vertex from S_k for $k < i$ with a color greater or equal to $w(S_i)$. Then the number of the required colors for achieving a proper coloring would increase by 1 because by the structure of the graph $(u, v) \in E(K_{s_1,...,s_n}, w)$ for $u \in S_k$ and $v \in S_i$. Thus, Alice's winning strategy illustrated above turns out to be optimal.

What is left is to show that $l + 2n - 3$ colors suffice for achieving a proper coloring of the entire graph if both players play their optimal strategies. We denote a vertex from the independent set S_i by v_{i_j} for $i \in \{1, ..., n\}$ and $j \in$

$\{1, ..., s_i\}$. Without loss of generality we can assume that throughout the game Alice colors vertex v_{i_1} and Bob colors vertex v_{i_2} where $v_{i_1}, v_{i_2} \in S_i$ for $i \in \{1, ..., n\}$. The following tabular demonstrates the run of the game considering the coloring of $\{S_n, ..., S_3\}$.

Player	Colored vertex	Color
Alice Bob	v_{n_1} v_{n_2}	$w(S_n)$ $l + n$
$\vdots$	$\vdots$	$\vdots$
Alice Bob	v_{i_1} v_{i_2}	$w(S_i)$ $l + n + (n - i)$
$\vdots$	$\vdots$	$\vdots$
Alice Bob	v_{3_1} v_{3_2}	$w(S_3)$ $l + 2n - 3$

Consider the point of the game where two unused colors $l, l+1$ and two independent sets $\{S_1, S_2\}$ with $S_i \cap C = \emptyset$ for $i \in \{1, 2\}$ remain. According to her winning strategy Alice colors vertex v_{2_1} with the color $l + 1 = w(S_2)$. Since $w(S_2) > w(S_1)$ Bob cannot assign to v_{2_2} the color l. Then by the structure of the graph the coloring of each independent set is fixed.

(ii) Suppose $w_{\max}(K_{s_1,...,s_n}, w)$ colors are given. By the same argument as above, we refer to Alice's optimal strategy in the previous case. Thus, she jumps "backwards" into the independent sets. However, Bob's strategy cannot run by the same method since only $w_{\max}(K_{s_1,...,s_n}, w)$ colors are given.

Bob's strategy: Assume that $S_n, S_{n-1}, ..., S_{j+1}$ contain colored vertices and it is Bob's turn. Obviously, he won't be able to attack $\{S_1, ..., S_j\}$ by coloring vertices from $\{S_n, S_{n-1}, ..., S_{j+1}\}$ since only $w_{\max}(K_{s_1,...,s_n}, w)$ colors are given. Thus, he attacks S_j by assigning to any vertex from the sets $S_1, S_2, ..., S_{j-1}$ the color $w(S_j)$. By the assumption $w(S_j) - w(S_{j-1}) \geq 2$, there is still an unused color for the vertices of S_j between $w(S_{j+1})$ and $w(S_j)$. Hence, $w_{\max}(K_{s_1,...,s_n}, w)$ colors suffice and Alice wins. $\square$

It is worth pointing out that for case (i) in proposition 3.2.7, there is another optimal strategy than σ from Bob's view which won't change $\gamma(K_{s_1,...,s_n}, w)$. If we admit the assumption $s_1 \geq n-2$, Bob could color vertices only in S_1 throughout the whole game and attack $\{S_2, ..., S_{n-1}\}$. Let us consider the following detailed exposition of the game from Bob's view; in particular it is clear that Alice's strategy does not vary from the one above according to her choice of the linear ordering L.

Bob's modified strategy is to stay in the set S_1 and to attack the highest weighted uncolored independent set, say S_q. With this strategy he forces Alice to use another color for coloring S_q. If he did not attack S_q, Alice could color a vertex in this set using the same color as its weight (which obviously would reduce the amount of colors required to color the graph). If Bob started in an arbitrary independent set, say S_i, he would only be able to attack the independent sets S_j with $j > i$. Since Alice jumps "backwards" in every step, Bob would also be forced sometimes to jump "back" either because every vertex in S_i is colored or because Alice jumps to an S_l with $l < i$ and hence Bob would lose an attack. Obviously, Bob is able to make the most attacks if he stays as long as possible in S_1 and then jumps to S_2 and so on. Since $s_1 \geq n-2$ he will be able to stay and color in S_1 throughout the whole game.

Let us now calculate how many colors are needed if both players play their optimal strategies. Alice starts coloring a vertex $v_n \in S_n$ with the color $w(S_n)$, while Bob colors a vertex in S_1 with the color $w(S_{n-1})$, which means that he attacks the uncolored independent set S_{n-1}. Alice goes on coloring S_{n-1} with $w(S_n)+1$, while Bob attacks the set S_{n-2} by coloring an arbitrary vertex in S_1 with $w(S_{n-2})$. This procedure lasts until every independent set contains at least one colored vertex, since after that point the game is trivial. See the following tabular:

Player	Step	Colored set	Color	Attacked set
Alice	1	S_n	$w(S_n)$	
Bob	2	S_1	$w(S_{n-1})$	S_{n-1}
Alice	3	S_{n-1}	$w(S_n)+1$	
Bob	4	S_1	$w(S_{n-2})$	S_{n-2}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Since Bob will produce $n-2$ attacks, we can conclude that

$$\gamma(K_{s_1,\ldots,s_n}, w) = w_{\max}(K_{s_1,\ldots,s_n}, w) + n - 2.$$

For the case $s_1 < n-2$ this method breaks down. If Bob starts coloring the vertices in S_1, there comes the point he has to jump to another independent set. One may ask in which independent set should Bob jump.

If he jumps to one of the independent sets Alice has already colored, the optimal he can apply from this point is to follow the strategy σ on page 45. Obviously, this reduces the amount of required colors by 1 since S_1 contains colored vertices.

If he jumps to an uncolored independent set, he saves this set from Alice's view and clearly this also reduces the amount of colors needed to guarantee a proper coloring. Thus, he would jump to S_2 because from there he is able to attack all the uncolored independent sets. If every vertex in S_2 is colored, he jumps to S_3 and so on. Jumping from S_1 to another uncolored independent set than S_2, say S_k, would not make sense for Bob because then two cases are possible:

- Since $w(S_k) > w(S_j)$ for $j \in \{k-1, ..., 2\}$ he is not able to attack the vertices from the sets $\{k-1, ..., 2\}$ and he has to jump back in order to be able to attack. Hence, he saves two independent sets from Alice view.

- If every vertex in S_k is colored, he has to jump to another uncolored independent set. In this case he saves one independent set as in the case he jumps to S_2.

Using this strategy, the amount of colors needed for a proper coloring reduces by the amount of jumps done by Bob in contrast to $\gamma(K_{s_1,\ldots,s_n}, w)$ since every jump makes one color redundant. It is obvious that Bob has to jump to S_2 iff $s_1 < (n-1)-1$. Further, it is also clear that he has to jump to S_3 if and only if $s_1 + s_2 < (n-1)-2$. In conclusion Bob will jump as long as $\sum_{i=1}^{t} s_i < (n-1)-t$. Hence, the maximum t for which $\sum_{i=1}^{t} s_i < (n-1)-t$ holds is the amount of jumps Bob will make.

In the previous work we dealt with weighted complete multipartite graphs considering that the vertex-weights of one independent set are equal. In the following we will generalize the distribution of the vertex-weights and analyze complete multipartite graphs with independent sets containing vertices with distinct weights.

Definition 3.2.8. A complete multipartite graph is called (k,l)-*weighted*, denoted by $(K^{(k,l)}_{s_1,\ldots,s_n}, w)$, iff every independent set assumes two distinct weights k and l with $k > l$.

Let $(K^{(k,l)}_{s_1,\ldots,s_n}, w) = (V, E, w)$ be a (k,l)-weighted complete multipartite graph. Then let

$$V^l = \{v \mid w(v) = l \text{ for } v \in V\}$$

and

$$V^k = \{v \mid w(v) = k \text{ for } v \in V\}$$

be the sets of all vertices with vertex-weights l and k, respectively. Further let $q = |V^l|$ and $p = |V^k|$ be the number of all vertices with vertex-weights l and k, respectively.

Proposition 3.2.9. *Let $(K^{(k,l)}_{s_1,\ldots,s_n}, w) = (V, E, w)$ be a (k,l)-weighted complete multipartite graph with $|\{v \mid w(v) = k \text{ and } v \in S_i\}| \geq 4$ for all $i \in \{1, \ldots, n\}$. If $k - l \geq q$, then*

$$\gamma(K^{(k,l)}_{s_1,\ldots,s_n}, w) = \begin{cases} k + 2n - 3, & \text{if } q \text{ odd,} \\ k + 2n - 2, & \text{else.} \end{cases}$$

Proof. By the structure of the graph, it is clear that the coloring of an independent set S_i is fixed as soon as an arbitrary vertex $v \in S_i$ is colored with a color greater or equal to k. Thus, Alice's optimal strategy is to force Bob to color first in an independent set with a color greater or equal k. However, Bob's optimal strategy is to prevent this. This implies that both color only vertices with vertex-weights l with feasible colors between l and $k-1$. Thus, until they reach the point of the game where one of them has to color a vertex with weight k first, their optimal strategies are the same which is the following:

Let C be the set of all colored vertices. Further, let $V^k \cap C = \emptyset$.

- If $V^l \not\subset C$, then color an arbitrary vertex from V^l with a feasible color between l and $k-1$.
- If $V^l \subset C$, then color an arbitrary vertex with a color greater or equal k.

The respective strategies change as soon as they come to the point of the game where $V^k \cap C \neq \emptyset$. Let $c(v)$ be the color of vertex v.

Alice's strategy:

- If there exists an i_0 such that $c(v) \leq k-1$ for all $v \in S_{i_0}$, then she colors an arbitrary vertex in S_{i_0} with the smallest feasible color.
- Otherwise, she colors vertices at random.

Assume that there exist i_0 and j_0 such that $c(v) \leq k-1$ for all $v \in S_{i_0}$ and there exists at least one vertex $u \in S_{j_0}$ such that $c(u) \geq k$. If Alice colored a vertex $u' \in S_{j_0}$, then Bob could color another vertex $u'' \in S_{j_0}$ with a color greater than k. This would imply that the number of the required colors for achieving a proper coloring of the graph would increase by 1 because $(u'', v') \in E(K^{(k,l)}_{s_1,\ldots,s_n}, w)$ for $v' \in S_{i_0}$. Thus, this strategy turns out to be optimal.

Bob's strategy:

- Assume that there exists an i_0 such that there is at least one vertex v with $c(v) \geq k$, then he colors an arbitrary vertex in S_{i_0} with the greatest feasible color.
- Otherwise, he colors at random.

If Bob colored a vertex $v' \in S_{i_0}$ with the color $c(v)$, which would be possible since $(v', v) \notin E(K^{(k,l)}_{s_1,...,s_n}, w)$, then the number of the required colors for a proper coloring of the graph would decrease by 1. Hence, the strategy illustrated above is the worst case scenario that can occur from Alice's view.

We are left with the task of determining the number of the required colors for achieving a proper coloring of the graph. By the optimal strategies of the players and by the assumption $k - l \geq q$, only colors between l and $k - 1$ are being assigned as soon as $V^l \subset C$. Let q be odd and assume that $k + 2n - 3$ colors are given. Then Bob is the one who is forced to color a vertex with weight k first. Without loss of generality, assume that he colors a vertex in S_1 with $k + 2n - 3$. Further, suppose that Alice colors along the linear ordering $\{S_2, ..., S_n\}$. Then by her strategy Alice jumps into S_2 and colors an arbitrary vertex. Due to Bob's strategy, he colors either in S_1 or in S_2 with $k + 2n - 4$, which is possible since $|\{v \mid w(v) = k \text{ and } v \in S_i\}| \geq 4$ for all $i \in \{1, ..., n\}$. They proceed until Alice color a vertex in S_n. Then the coloring of the graph is fixed. Finally, we can conclude that each independent set from $\{S_1, ..., S_n\}$ is assigned to two distinct colors greater or equal k, except S_1 and S_n. Hence, $k - 1 + 2n - 2$ colors suffice for achieving a proper coloring. Thus,

$$\gamma(K^{(k,l)}_{s_1,...,s_n}, w) = k + 2n - 3.$$

If we assume that q is even and $k + 2n - 2$ colors are given, then Alice is the one who has to color first a vertex with weight k. Without loss of generality assume that she colors in S_1 with the color k. Then by the assumption $|\{v \mid w(v) = k \text{ and } v \in S_i\}| \geq 2$ for all $i \in \{1, ..., n\}$, Bob colors in S_1 with $k + 2n - 2$. Suppose that Alice colors along the linear ordering $\{S_1, ..., S_n\}$. Then each independent set from $\{S_1, ..., S_n\}$ is assigned to two distinct colors greater or equal k, except S_n. Hence, we can conclude that

$$\gamma(K^{(k,l)}_{s_1,...,s_n}, w) = k - 1 + 2n - 1 = k + 2n - 2. \qquad \square$$

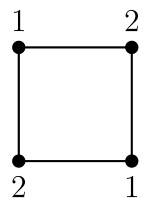
Remark 3.2.10. Let q be odd. Then according to the proposition above, Bob is the one who colors a vertex with weight k first as soon as all vertices with weight l are colored. One could conjecture that $\gamma(K^{(k,l)}_{s_1,...,s_n}, w) = \chi(K^{(k,l)}_{s_1,...,s_n}, w)$

see definition 1.2.3, if we assume that $|\{v \mid w(v) = k \text{ and } v \in S_i\}| = 2$ for all $i \in \{1, ..., n\}$ because of the following consideration. Suppose in his last turn Bob colored the first vertex with weight k. Without loss of generality assume that he colored vertex $v \in S_i$ and it is Alice's turn now. Then Alice could color $v' \in S_i$ with the color as v, which is possible since $(v', v) \notin E(K^{(k,l)}_{s_1,...,s_n}, w)$. Then by the assumption that each independent set contains only 2 vertices with weight k and because each vertex with weight l is already colored, Bob would be forced to jump into another independent set S_j where $c(v) \leq k-1$ for all $v \in S_j$ and to color one of the vertices with weight k with a color greater or equal k first. Obviously, Alice would proceed as in her last turn. Thus, the number of the required colors would equal to the trivial lower bound of $\gamma(K^{(k,l)}_{s_1,...,s_n}, w)$ which is $k - 1 + n$. However, Bob would apply strategy σ introduced on page 45 from the beginning.

Once in an independent set S_i a vertex has been colored with a color greater or equal k, the coloring of S_i is fixed. Thus, σ ensures that each time Bob takes turn, Alice has to jump into a new set S_j and fix the coloring of S_j with a feasible color greater or equal k. This implies that each independent set from $\{S_1, ..., S_n\}$ is assigned to 2 distinct colors except S_1 and S_n, assuming that Bob assigned to a vertex from S_1 the color k. Then we can conclude that $k - 1 + 2n - 2 = k + 2n - 3$ color are sufficient for achieving a proper coloring of the graph.

3.3 The Game Chromatic Number of Weighted Cycles

In the following we will analyze the game chromatic number of weighted cycles. It is to be expected that the game chromatic number of weighted cycles is greater or equal to the game chromatic number of cycles without vertex-weights. However, we will figure out a surprising result. It is well known that the chromatic number of a weighted graph $\chi(G, w)$ is greater or equal to the ordinary chromatic number since each weighted vertex is being colored with a color greater or equal to its weight, which is equal to 1 in an ordinary graph without vertex weights. One could also conjecture that this holds considering the game versions of these chromatic numbers, that would be $\gamma(G, w) \geq \gamma(G)$. In fact this inequality does not hold. Consider a cycle of length 4. It is well known that 3 is the game chromatic number of this graph. Consider a cycle of length 4 with weights 1 and 2 so that the weights are assumed alternating (see the following figure).



The game chromatic number of this graph is 2. Suppose 2 colors are given. If Alice starts coloring a vertex of weight 2 with color 2, then the outcome of the game is trivial: Bob won't be able to attack any vertex. He has to color the vertices with the color of their own vertex-weights. Hence, the game chromatic number of the weighted cycle above is less than the ordinary game chromatic number. In section 3.4 we are going to construct graphs with this property.

In this section it is our interest to find out how many colors are required in order to achieve a proper coloring of a vertex-weighted cycle. First we will discuss a general estimation of $\gamma(C, w)$ and prove that it is at most $w_{\max}(C, w) + 2$. Furthermore, the reader will be familiarized with the game chromatic number of a cycle with a certain distribution of the vertex-weights. Thus, in proposition 3.3.3 we will consider $(k_1, ..., k_m)$-*alternating-weighted* cycles, where for each two vertices x_i and x_j it holds $w(x_i) = w(x_j)$ whenever $d_{(x_i, x_j)} = m$.

Throughout the section we will take advantage of the structure of a cycle, which implies that every vertex has exactly two neighbors.

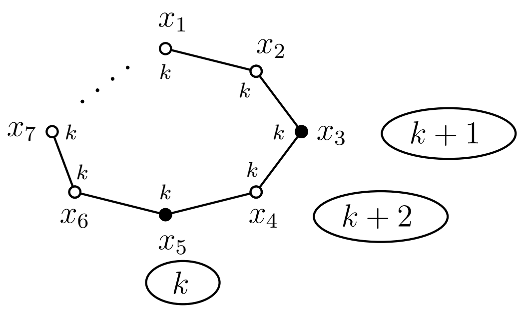
Proposition 3.3.1. *Let* $(C, w) = (V, E, w)$ *a weighted cycle. Then it holds*

$$w_{\max}(C, w) \leq \gamma(C, w) \leq w_{\max}(C, w) + 2.$$

Proof. It is easily seen that the inequality $w_{\max}(C, w) \leq \gamma(C, w)$ holds. It remains to show that $\gamma(C, w) \leq w_{\max}(C, w) + 2$. Let $|V| = n$. Because of the elementary structure of a cycle it is sufficient to consider the sequence $\{(x_1, x_2, x_3), (x_2, x_3, x_4), ..., (x_n, x_1, x_2)\}$ where $x_i \in V$ for $i \in \{1, ..., n\}$. Hence, each vertex $x_j \in V$ can be attacked by Bob at most twice by coloring its neighbors with $w(x_j)$ and $w(x_j) + 1$, if possible. One will need a third color in order to achieve a proper coloring. Hence, the worst case that can occur is , if $w(x_j) = w_{\max}(C, w)$. Since $|\{w_{\max}(C, w), w_{\max}(C, w) + 1, w_{\max}(C, w) + 2\}| = 3$ Alice's victory is guaranteed, independent of the strategies of the players. $\square$

Example:

(i) Consider the cycle $(C, w) = (V, E, w)$ with $|V| = n$ and $w(x_1) = w(x_2) = ... = w(x_n) = k$. Assume $k + 2 = w_{\max}(C, w) + 2$ colors are given and Alice colors vertex x_5 with the color k. Further Bob assigns to vertex x_3 the color $k + 1$. Hence, vertex x_4 is attacked twice. Since there is still a third color $k + 2$ left for x_4 the graph can be colored with the given amount of colors. Thus, Alice wins.

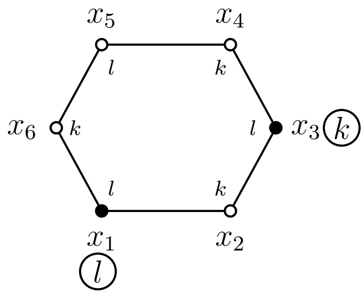


(ii) Consider the cycle $(C, w) = (V, E, w)$ with $|V| = n$. Further assume that the vertices of (C, w) obtain alternating two different vertex-weights k and l, where $k > l$ and $n \bmod 2 = 0$. Then we have

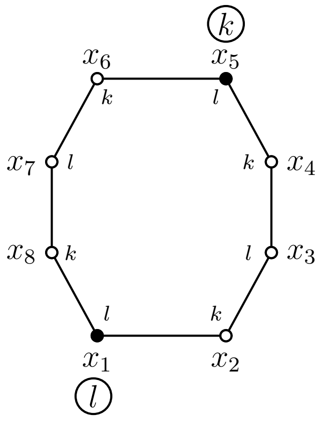
$$\gamma(C, w) = \begin{cases} k, & \text{if } n = 4, \\ k + 1, & \text{if } n = 6, \\ k + 2, & \text{if } n \geq 8. \end{cases}$$

If $n = 4$, we can refer to the example in the beginning of the section.

Further, we consider Alice's winning strategy for $n = 6$. Without loss of generality she starts the game by coloring vertex x_1 with the color l. Since Bob's purpose is to prevent a proper coloring with the given amount of colors, he will try to assign x_{j-1} and x_{j+1} the colors k and $k+1$, where $w(x_j) = k$ for $j \in \{1, ..., 6\}$. Thus, it is easily seen that Bob will color either x_3 or x_5 with the color k in order to attack vertex x_4. Assume he goes for x_3. Then Alice has to assign to vertex x_4 the color $k+1$ or to vertex x_5 the color l or k, immediately. Otherwise, Bob would color x_5 with $k+1$ in his next step.



Finally, we are left with the task of determining $\gamma(C, w)$ for $n \geq 8$. Assume that $k+2$ colors are given and Alice starts the game by coloring vertex x_1 with the color l. Obviously, Bob colors either vertex x_5 in order to attack vertices x_4 and x_6 or vertex x_7 in order to attack vertices x_6 and x_8 with the color k. Assume he goes for x_5: If Alice colors x_4 with the color $k+1$, Bob would color x_7 with $k+1$ or $k+2$. This implies that vertex x_6 is being attacked twice. However, there is still a color left for x_6. The same conclusion can be drawn for the case that Alice colors vertex x_6.



Definition 3.3.2. Let $m \leq n$ for $m \in \mathbb{N}$ and $(C, w) = (V, E, w)$ be a cycle with $|V| = n$ and $n \bmod m = 0$. We call $(C, w) = (V, E, w)$ $(k_1, ..., k_m)$*-alternating-weighted*, if $w : V \to \{k_1, ..., k_m\}$ with $w(x_{i+m \times j}) = k_i$, for $i = \{1, ..., m\}$ and

$j \in \mathbb{N}$.

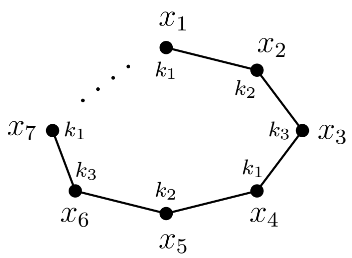


FIGURE A (k_1, k_2, k_3)-alternating-weighted cycle

Proposition 3.3.3. *Let $(C, w) = (V, E, w)$ be a $(k_1, ..., k_m)$-alternating-weighted cycle with $k_1 < k_2 < ... < k_m$ and $m \geq 3$. Then*

$$\gamma(C, w) = w_{\max}(C, w) + 1.$$

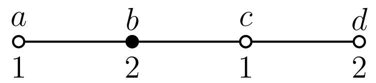
Proof. Because of the structure of a cycle we can conclude how often a vertex v can be attacked by Bob by coloring a neighboring vertex with $w(v)$, taking into account that the assigned color is at least the vertex-weight. Without loss of generality we will consider the path $\{x_{n-1}, x_n, x_1\}$ with vertex-weights $k_{m-1} < k_m > k_1$:

- Vertex x_1 cannot be attacked since $k_1 < k_m$ and $k_1 < k_2$.
- If $k_{m-1} + 1 = k_m$, vertex x_{n-1} can be attacked twice by coloring x_{n-2} with k_{m-1} and x_n with k_m. However, a proper coloring is still possible since vertex x_{n-1} can be colored with $w_{\max}(C, w) + 1$. Otherwise, if $k_{m-1} + 1 < k_m$, it can be attacked only once.
- Vertex x_n can be attacked twice since $k_{m-1} < k_m$ and $k_1 < k_m$.

It is obvious that the third case illustrates the worst case, where Bob assigns to a neighbor of an arbitrary vertex x_p with vertex-weight k_m the color k_m. However, Alice's winning strategy is to prevent that a vertex x_p with $x_p = k_m$ is being attacked twice by following him and coloring x_p with the color $k_m + 1$. Otherwise, Bob could color the other neighbor of x_p with $k_m + 1$, which would implicate that a proper coloring with $k_m + 1$ colors won't be possible anymore. $\square$

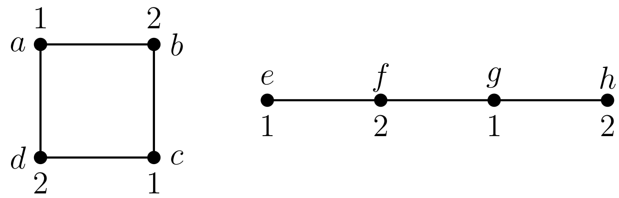
3.4 Construction of Graphs with $\gamma(\mathbf{G}, \mathbf{w}) < \gamma(\mathbf{G})$

In the previous section we showed that there exists a weighted cycle (C, w) such that $\gamma(C, w) < \gamma(C)$. The aim of this section is to construct more graphs with this property. Throughout the section we will denote a weighted graph by $(G, w) = (V, E, w)$ and by $G = (V, E)$ the corresponding graph without weights. Moreover, we will consider connected graphs. Consider the path (P_4, w) of length 3 below.



While the game chromatic number of a path with 4 vertices without vertex-weights is 3, it holds that $\gamma(P_4, w) = 2$: Alice starts the game by coloring vertex b with the color 2 and the coloring is fixed. Based on the (P_4, w) and the cycle (C_4, w) on page 66 we will construct more graphs with $\gamma(G, w) < \gamma(G)$. Moreover, consider all possible graphs on 3 vertices. Then we have either a K_3 or a P_3. Obviously, $\gamma(K_3, w) \geq \gamma(K_3)$ and $\gamma(P_3, w) \geq \gamma(P_3)$ holds no matter which weight distribution we take into consideration. Hence, a graph (G, w) with $\gamma(G, w) < \gamma(G)$ must have at least 4 vertices.

Consider the (C_4, w) and the (P_4, w) with alternating vertex-weights where 1 and 2 are the allowed weights.



For the rest of this section we will denote the graphs above by $C_4^{(1,2)}$ and $P_4^{(1,2)}$ and will call a vertex *key-vertex*, if it is adjacent to every vertex with weight 1. Vertices b, d and f are key vertices. Now consider the following operation on either $C_4^{(1,2)}$ or $P_4^{(1,2)}$.

Operation I: Add edges and vertices so that vertices with weight 1 are only adjacent to vertices with weight 2 and vice versa. Choose at least one vertex with weight 2 and make it a key-vertex.

Lemma 3.4.1. *Let $(G, w) = (V, E, w)$ be a weighted graph which resulted by applying operation I on either $C_4^{(1,2)}$ or $P_4^{(1,2)}$. Then $\gamma(G, w) = 2$ and $\gamma(G) \geq 3$.*

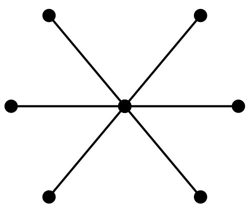
Proof. Let $(G, w) = (V, E, w)$ be a weighted graph which resulted by applying operation I on either $C_4^{(1,2)}$ or $P_4^{(1,2)}$. Assume that 2 colors are given. If Alice colors a key-vertex with the color 2, the coloring is fixed since every vertex with weight 1 is adjacent to the key-vertex and has only neighbors with weight 2. Hence, vertices with weight 1 cannot be attacked by Bob. Vertices with weight 2 also cannot be attacked since they are only adjacent to vertices with weight 1 that cannot be colored with the color 2 because they are adjacent to the already colored key-vertex.

$\gamma(G) \geq 3$ is clear, since our graph G contains either an induced C_4 or P_4. $\square$

Now we go a step further and investigate the class of graphs that satisfy the property $\gamma(G, w) = 2$ and $\gamma(G) \geq 3$. For this purpose we need to give the definition of the class of stars.

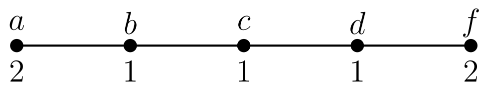
Definition 3.4.2. A graph $G = (V, E)$ with vertex set $V = \{v_1, v_2, ..., v_n\}$ for $n \in \mathbb{N}$ is called a *star*, if there exists one and only one $i_0 \in \{1, ..., n\}$ such that $(v_{i_0}, v_j) \in E$ and $d(v_j) = 1$ for all $j \in \{1, ..., n\} \setminus \{i_0\}$.

Example:



For a star we can conclude the trivial condition that for the distance between two vertices it holds $d_{(v_{i_0}, v_j)} = 1$ and $d_{(v_j, v_{j'})} = 2$ for all $j, j' \in \{1, ..., n\} \setminus \{i_0\}$. Hence, the class of stars has game chromatic number equal to 2, since Alice just has to color vertex v_{i_0} in her first move which fixes the coloring.

Consider further the following graph (G, w).



Assume that 2 colors are given. Since c is a key-vertex, Alice has to color c with 2 in order to win the game. This example demonstrates that vertices with weight 1 can also be adjacent to each other, while $2 = \gamma(G, w) < \gamma(G)$ holds. Obviously, this is only possible if (G, w) is a bipartite graph and as long as at least one key-vertex exists. The condition of the existence of a key-vertex implies that if the key-vertex has weight 2 there cannot exist two adjacent vertices with weight 1, because this would produce a K_3. If the key-vertex has weight 1, there cannot exist a (P_4, w) whose vertices have weight 1. Moreover, if there exists a (P_3, w) whose vertices have weight 1, then there are no other two adjacent vertices with weight 1. The same holds if there exists a (P_2, w) whose vertices have weight 1.

We say that a weighted bipartite graph $(G, w) = (V, E, w)$ with $|V(G, w)| \geq 4$ is an element of the class $\mathcal{Q}_w$ if (G, w) satisfies the following conditions:

- (G, w) is not a star.
- (G, w) has vertex-weights at most 2. Vertices with weight 2 are not adjacent and vertices with weight 1 are allowed to be adjacent.
- (G, w) has at least one key-vertex x and $w(y) = 1$, if $y \in N(x)$.

Obviously, $\gamma(G, w) = 2$ for $(G, w) \in \mathcal{Q}_w$ since Alice needs to color the key vertex with 2 in order to fix the coloring of the graph and $\gamma(G) \geq 3$ for the corresponding graph without weights. Moreover, the class of graphs that can be constructed by operation I from $C_4^{w(1,2)}$ or $P_4^{w(1,2)}$ is contained in $\mathcal{Q}_w$.

We want to investigate, if there exist more graphs with $\gamma(G, w) = 2$ and $\gamma(G) \geq 3$. The answer is no and the next theorem proves this claim.

Theorem 3.4.3. $(G, w) = (V, E, w)$ *is a weighted connected graph with* $\gamma(G, w) = 2$ *and* $\gamma(G) \geq 3$ *if and only if* $(G, w) \in \mathcal{Q}_w$.

Proof. We already know that if $(G, w) \in \mathcal{Q}_w$, then $\gamma(G, w) = 2$ and $\gamma(G) \geq 3$.

Assume $(G, w) = (V, E, w)$ is a weighted graph with $\gamma(G, w) = 2$ and with $\gamma(G) \geq 3$ so that $(G, w) \notin \mathcal{Q}_w$. Further we can assume that there exists no vertex in (G, w) with weight greater or equal to 3 since $\gamma(G, w) = 2$. Moreover, in (G, w) there does not exist 2 vertices $x, y \in V(G, w)$ with $w(x) = w(y) = 2$ and $(x, y) \in E(G, w)$ since this would mean that $\gamma(G, w) \geq 3$ contradicting our assumption. Moreover, (G, w) is bipartite, (G, w) is not a star and as already shown $|V| \geq 4$ must hold.

Assume further that there exists no key-vertex. Suppose that Alice starts the game coloring a vertex x with weight 2. Then there exists a vertex y with weight 1 so that $(x, y) \notin E(G, w)$ and a shortest path $P = x, v_1, ..., v_n, y$ from x to y. If y has a neighbor with weight 2, Bob colors y with 2 and wins the game. If every neighbor of y has weight 1, Bob considers vertex v_n. If v_n has a neighbor with weight 2, Bob colors v_n with 2 and wins the game. Otherwise, he jumps backwards until he reaches a vertex v_i for $i \in \{2, ..., n-1\}$ which is adjacent to a vertex with weight 2. If Bob reaches v_2 without finding a vertex which is adjacent to a vertex with weight 2, he does the following. He colors v_2 with 1 (this is possible since v_3 has no neighbors with weight 2) and he wins the game since x is colored with 2 and there is no available color left for v_1. With the same procedure we can show that Bob wins the game, if Alice decides to color a vertex with weight 1 first.

Hence, $\gamma(G, w) \geq 3$, contradicting our initial assumption. In conclusion, $(G, w) \in \mathcal{Q}_w$. □

3.5 The Game Chromatic Number of Weighted Trees

In this section we will determine the game chromatic number of weighted trees. We will apply an algorithm which is due to Faigle, Kern, Kierstead and Trotter in [14], illustrated in the appendix A.2 for a better understanding. They proved that $\gamma(T) \leq 4$ where T is tree without vertex-weights; in particular, they estimated the maximum number of colored adjacent vertices of an uncolored vertex during the game by 3. We will make use of this fact and additionally for the case of weighted graphs we need to observe the so called *end-vertices* and *interior-vertices* of a tree, which we will define below.

Definition 3.5.1. Let $(T, w) = (V, E, w)$ be a weighted tree. A vertex $v \in V$ with $d(v) = 1$ is called *end-vertex* or *leaf*. We denote the set of the end-vertices by $\overline{V}$ and by $\overline{w}_{\max}(T, w)$ the maximum weight of the end-vertices. A vertex $w \in V$ with $d(w) \geq 2$ is called *interior-vertex*. We denote the set of the interior-vertices by $\mathring{V}$ and by $\mathring{w}_{\max}(T, w)$ the maximum weight of the interior-vertices.

Proposition 3.5.2. *Let $(T, w) = (V, E, w)$ be a weighted tree. Then the following holds.*

$$\gamma(T, w) \leq \begin{cases} \max\{\overline{w}_{\max}(T, w) + 1, \mathring{w}_{\max}(T, w) + 3\}, & \text{if there exists a } v \text{ with} \\ & \quad w(v) = \mathring{w}_{\max}(T, w) \wedge d(v) \geq 3, \\ \max\{\overline{w}_{\max}(T, w) + 1, \mathring{w}_{\max}(T, w) + 2\}, & \text{else.} \end{cases}$$

Proof. The proof will be divided into two parts by analyzing the coloring of the interior- and end-vertices separately:

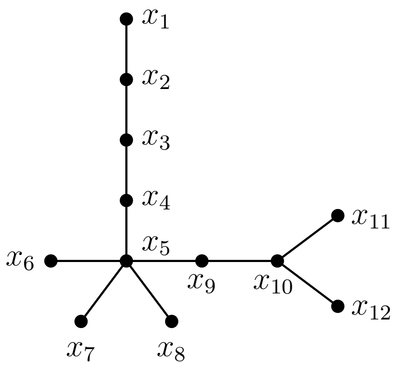
Due to Kierstead's algorithm, the worst case takes place, if Bob achieves to color three neighbors $\{v_1, v_2, v_3\}$ of an uncolored vertex v with three distinct colors, provided that v has three neighbors. Hence, the worst case for weighted graphs is assumed if $w(v) = \mathring{w}_{\max}(T, w)$ and $\{v_1, v_2, v_3\}$ are colored with $\{\mathring{w}_{\max}(T, w), \mathring{w}_{\max}(T, w) + 1, \mathring{w}_{\max}(T, w) + 2\}$. Thus, we can draw the conclusion that $\mathring{w}_{\max}(T, w) + 3$ colors are sufficient for coloring all interior vertices, if there exists a vertex $v \in \mathring{V}(T, w)$ with $w(v) = \mathring{w}_{\max}(T, w)$ and $d(v) \geq 3$. Otherwise, since $d(v) \geq 2$ for all $v \in \mathring{V}$, $\mathring{w}_{\max}(T, w) + 2$ colors are sufficient for a proper coloring of $\mathring{V}$.

Now we take the end-vertices under consideration. Obviously, if $\overline{w}_{\max}(T, w) \leq \mathring{w}_{\max}(T, w)$, then $\gamma(T, w) \leq \mathring{w}_{\max}(T, w) + 3$. Suppose that $\overline{w}_{\max}(T, w) > \mathring{w}_{\max}(T, w)$. If $\overline{w}_{\max}(T, w) - \mathring{w}_{\max}(T, w) < 3$, then obviously $\gamma(T, w) \leq \mathring{w}_{max}(T, w) + 3$. Let $\overline{w}_{\max}(T, w) - \mathring{w}_{\max}(T, w) \geq 3$.

- If $\overline{w}_{\max}(T, w)$ is assumed only once with $w(v') = \overline{w}_{\max}(T, w)$, then at the beginning of the game Alice would color v' such that Bob won't be able to attack v' anymore. In this case $\overline{w}_{\max}(T, w)$ colors would be sufficient for achieving a proper coloring of (T, w).

- If $\overline{w}_{\max}(T,w)$ is assumed more than once, then Bob would be able to attack a vertex with weight $\overline{w}_{\max}(T,w)$ by giving its neighbor the color $\overline{w}_{\max}(T,w)$. Since $d(u) = 1$ for $u \in \overline{V}(T,w)$, vertex u can be attacked only once. Thus, $\overline{w}_{\max}(T,w)+1$ colors are required. $\square$

Example:



For the tree above we have $\overline{V}(T,w) = \{x_1, x_6, x_7, x_8, x_{11}, x_{12}\}$ and $\mathring{V}(T,w) = \{x_2, x_3, x_4, x_5, x_9, x_{10}\}$. Let $w(x_5) = \mathring{w}_{\max}(T,w)$ and $w(x_{11}) = \overline{w}_{\max}(T,w)$.

(i) Let $\mathring{w}_{\max}(T,w) = k$ and $\overline{w}(T,w) = k+1$. Then a proper coloring with $k+3$ given colors should be possible. See the following procedure of the game where Alice's winning strategy is based on Kierstead's algorithm. Let $c(x_i)$ be the color of vertex x_i for $i \in \{1,...,12\}$.

Player	Turn	Vertex	Color	
Alice	1	x_1	$w(x_1)$	
Bob	1	x_6	$w(x_5) = k$	$\rightarrow$ 1^{st} attack on vertex x_5.
Alice	2	x_2	$w(x_2)$, if $w(x_2) \neq c(x_1)$; $c(x_1)+1$ else	
Bob	2	x_4	$k+1$	$\rightarrow$ 2^{nd} attack on vertex x_5.
Alice	3	x_3	$w(x_3)$, if $w(x_3) \neq c(x_2)$ $k+2$, else	
Bob	3	x_7	$k+2$	$\rightarrow$ 3^{rd} attack on vertex x_5.
Alice	4	x_5	$k+3$	

The rest of the game is trivial.

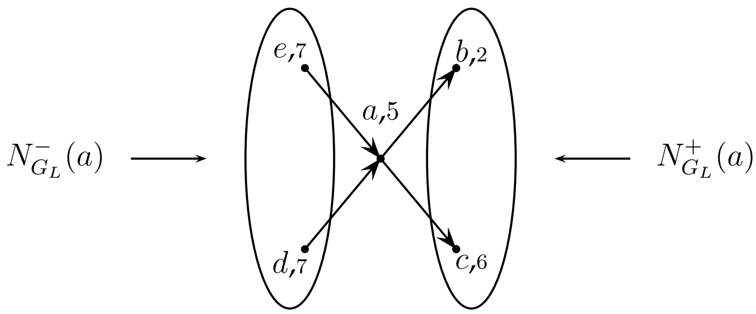
(ii) Let $\mathring{w}_{\max}(T, w) = k$ and $\overline{w}_{\max}(T, w) = k + 3$ and suppose that $w(x_8) = w(x_{11}) = k + 3$. Then a proper coloring of (T, w) with $k + 4$ colors should be possible. Since $\overline{w}_{\max}(T, w)$ is assigned to two vertices from $\overline{V}$, in his first turn Bob will be able to attack either x_8 or x_{11}, by coloring x_5 or x_{10} with $k+3$. Since x_8 and x_{11} are end-vertices, they can be attacked only once and hence $k + 4$ colors suffice.

3.6 The Game Chromatic Number of Weighted Planar Graphs

In this section we intend to give an upper bound for the game chromatic number of weighted planar graphs. As in section 2.4 (*Circular Game Chromatic Number of Planar Graphs*) we will make use of Kierstead's algorithm, introduced in [11], by adapting the *marking game on graphs* and the *rank of a graph* for the case of weighted graphs. In particular, we refer to a weighted graph $(G, w) = (V, E, w)$ with $|V| = n$ and follow the notations on page 26 regarding the linear ordering L on $V(G, w)$, the orientation (G_L, w) and the in- and outneighborhood $(N^+_{G_L}, N^-_{G_L})$. For simplicity, we will denote a weighted graph (G, w) by G throughout this section. Furthermore, we need to introduce some new notations:

- For a vertex $v \in V(G)$ we define the *upper-outneighborhood* of v in G_L as the set of vertices from $N^+_{G_L}(v)$ with weight greater or equal to the weight of v. We denote it by $\hat{N}^+_{G_L}(v)$. The *under-outneighborhood* is defined to be the set of vertices from $N^+_{G_L}(v)$ with weight lower to the weight of v and is denoted by $\check{N}^+_{G_L}(v)$. The definitions of the *upper-inneighborhood* and the *under-inneighborhood* denoted by $\hat{N}^-_{G_L}(v)$ and $\check{N}^-_{G_L}(v)$, respectively, are analogue. The various degrees of v are denoted by $\hat{d}^+_{G_L}(v) = |\hat{N}^+_{G_L}(v)|$, $\check{d}^+_{G_L}(v) = |\check{N}^+_{G_L}(v)|$, $\hat{d}^-_{G_L}(v) = |\hat{N}^-_{G_L}(v)|$ and $\check{d}^-_{G_L}(v) = |\check{N}^-_{G_L}(v)|$. The *maximum upper-outdegree* of G_L is denoted by $\hat{\Delta}^+_{G_L}$ and the *maximum under-outdegree* by $\check{\Delta}^+_{G_L}$. $\hat{\Delta}^-_{G_L}$ and $\check{\Delta}^-_{G_L}$ are defined analogue.

Example: Consider the graph below with vertices $\{a, b, c, d, e\}$, where the numbers indicate the vertex weights. Obviously, $\hat{d}^+_{G_L}(a) = \check{d}^+_{G_L}(a) = 1$, $\hat{d}^-_{G_L}(a) = 2$ and $\check{d}^-_{G_L}(a) = 0$ with $\hat{N}^+_{G_L}(a) = \{c\}$, $\check{N}^+_{G_L}(a) = \{b\}$, $\hat{N}^-_{G_L}(a) = N^-_{G_L}(a) = \{e, d\}$ and $\check{N}^-_{G_L}(a) = \emptyset$.



Furthermore, we introduce the $H^+_{G_L}$*-outneighborhood* of a vertex $v \in V(G)$ as follows:

- Set $\mathtt{g(v) = w(v)}$;
- if $(|\{\mathtt{x} \in \mathtt{N}^+_{\mathtt{G_L}}(\mathtt{v}) \mid \mathtt{w(x)} \leq \mathtt{g(v)}\}| > 0)$

$$\texttt{Set } \mathtt{h(v)} = \mathtt{g(v)} + |\{\mathtt{x} \in \mathtt{N}^+_{\mathtt{G_L}}(\mathtt{v}) \mid \mathtt{w(x)} \leq \mathtt{g(v)}\}|;$$

$$\texttt{while } (\{\mathtt{x} \in \mathtt{N}^+_{\mathtt{G_L}}(\mathtt{v}) \mid \mathtt{w(x)} \leq \mathtt{g(v)}\} \neq \{\mathtt{x} \in \mathtt{N}^+_{\mathtt{G_L}}(\mathtt{v}) \mid \mathtt{w(x)} \leq \mathtt{h(v)}\})$$

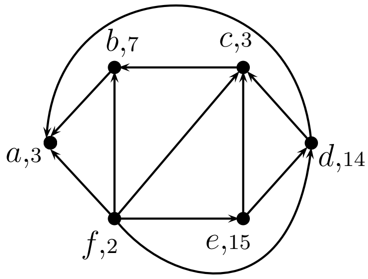
$$\begin{aligned} \texttt{Update} \quad \mathtt{h(v)} &= \mathtt{h(v)} + |\{\mathtt{x} \in \mathtt{N}^+_{\mathtt{G_L}}(\mathtt{v}) | \mathtt{w(x)} \leq \mathtt{h(v)}\}| \\ &\quad - |\{\mathtt{x} \in \mathtt{N}^+_{\mathtt{G_L}}(\mathtt{v}) | \mathtt{w(x)} \leq \mathtt{g(v)}\}|; \\ \texttt{Update} \quad \mathtt{g(v)} &= \mathtt{h(v)} - |\{\mathtt{x} \in \mathtt{N}^+_{\mathtt{G_L}}(\mathtt{v}) | \mathtt{w(x)} \leq \mathtt{h(v)}\}| \\ &\quad - |\{\mathtt{x} \in \mathtt{N}^+_{\mathtt{G_L}}(\mathtt{v}) | \mathtt{w(x)} \leq \mathtt{g(v)}\}|; \end{aligned}$$

else $\mathtt{h(v) = w(v)}$;

The outcome of this algorithm is the positive integer $h(v)$. We define the $H^+_{G_L}$-outneighborhood of $v \in V(G)$ as the set of vertices $x \in N^+_{G_L}(v)$ with $w(x) \leq h(v)$ and denote it by $H^+_{G_L}(v)$. We set $d^{H^+_{G_L}}(v) = |H^+_{G_L}(v)|$ and denote the maximum $H^+_{G_L}$-outneighborhood of G_L by $\Delta^{H^+_{G_L}}$. Further, we define

$$\bar{\Delta}^{H^+_{G_L}} := \max\{h(x) : x \in V(G)\}$$

Example: Consider the following graph with $L = \{a, b, c, d, e, f\}$:



Vertex	$\hat{N}^+_{G_L}(\cdot)$	$\hat{N}^-_{G_L}(\cdot)$	$\check{N}^+_{G_L}(\cdot)$	$\check{N}^-_{G_L}(\cdot)$	$h(\cdot)$	$H^+_{G_L}(\cdot)$
a	$\{\emptyset\}$	$\{b, d\}$	$\{\emptyset\}$	$\{f\}$	3	$\{\emptyset\}$
b	$\{\emptyset\}$	$\{\emptyset\}$	$\{a\}$	$\{c, f\}$	8	$\{a\}$
c	$\{b\}$	$\{d, e\}$	$\{\emptyset\}$	$\{f\}$	3	$\{\emptyset\}$
d	$\{\emptyset\}$	$\{e\}$	$\{a, c\}$	$\{\emptyset\}$	16	$\{a, c\}$
e	$\{\emptyset\}$	$\{\emptyset\}$	$\{c, d\}$	$\{\emptyset\}$	17	$\{c, d\}$
f	$\{a, b, c, d, e\}$	$\{\emptyset\}$	$\{\emptyset\}$	$\{\emptyset\}$	2	$\{\emptyset\}$

For the graph above we have $\bar{\Delta}^{H^+_{G_L}} = 17$.

The Weighted Marking Game

Let G be a weighted graph and let t be a given integer. The *weighted marking game* is played on G by Alice and Bob with Alice moving first. In each move the players take turns choosing vertices from the shrinking set $U \in V(G)$ of unchosen vertices (obviously $U = V(G)$ before the game starts). The result is a linear ordering $L \in \Pi(G)$ of the vertices of G with $x < y$ iff x is chosen before y. The *w-score* of the game is equal to $\bar{\Delta}^{H^+_{G_L}}$. Alice wins, if the w-score is at most the given integer t; otherwise, Bob wins.

The *weighted marking game number $col^w_g(G)$* of G is the least integer t such that Alice has a winning strategy for the weighted marking game, that is $\bar{\Delta}^{H^+_{G_L}} \leq t$.

Lemma 3.6.1. *For a weighted graph G we have $\gamma(G) \leq col^w_g(G)$.*

Proof. Suppose $col_g^w(G) = t$ and let S be the optimal strategy for Alice. If t colors are given and if Alice follows the strategy S and colors every vertex which is to be chosen with the smallest possible color, she wins also the coloring game. The reason lies in the definition of the w-score whose amount guarantees Alice always a free color. □

As mentioned above we use Kierstead's algorithm (*the activation strategy*) for our purposes. The algorithm can be found on page 28. We will use the strategy $S(L, G)$ without any change. Kierstead also introduced a parameter called the rank of a graph G. He achieved to restrict the marking game number with this parameter. For the case of weighted graphs we need to expand the definition of the rank of a graph G and will introduce the *weighted rank* of a weighted graph. Our aim is to restrict the weighted marking game number with this new parameter.

The Weighted Rank of a Weighted Graph

$$r^w(v, L, G) := w(v) + d^{H^+_{G_L}}(v) + m(v, L, G)$$

$$r^w(L, G) := \max_{v \in V(G)} r^w(v, L, G)$$

$$r^w(G) := \min_{L \in \Pi(G)} r^w(L, G)$$

The next proposition shows the relation between the weighted game marking number and the weighted rank of a weighted graph G.

Proposition 3.6.2. *For any weighted graph G and any ordering $L \in \Pi(G)$, if Alice uses the strategy $S(L, G)$ in order to play the weighted marking game on G, then the w-score will be at most $r^w(L, G)$. In particular, $col_g^w(G) \leq r^w(G)$.*

Proof. Suppose that Alice uses the strategy $S(L, G)$ to play the weighted marking game on G for an arbitrary $L \in \Pi(G)$. Let A denote the set of active vertices during the game. Since every vertex chosen by Bob immediately becomes active and any vertex chosen by Alice is already active, it remains to be shown that at any time t any unchosen vertex v is adjacent to at most

$d^{H^+_{G_L}}(v) + m(v, L, G)$ active vertices of the set $H^+_{G_L}(v)$, that is

$$|N(v) \cap A| - |\{x \,|\, x \in N^+(v) \cap A \wedge x \notin H^+_{G_L}(v)\}| \leq d^{H^+_{G_L}}(v) + m(v, L, G).$$

Since Kierstead proved in [11] that the inequality

$$|N(v) \cap A| \leq d^+_{G_L}(v) + m(v, L, G)$$

holds, we are able to conclude that

$$|N(v) \cap A| - |\{x \,|\, x \in N^+(v) \cap A \wedge x \notin H^+_{G_L}(v)\}| \leq d^{H^+_{G_L}}(v) + m(v, L, G).$$

Thus, if C denotes the set of chosen vertices at the time t, we have

$$\begin{aligned} &|N(v) \cap C| - |\{x \,|\, x \in N^+(v) \cap A \wedge x \notin H^+_{G_L}(v)\}| + w(v) \leq \\ &|N(v) \cap A| - |\{x \,|\, x \in N^+(v) \cap A \wedge x \notin H^+_{G_L}(v)\}| + w(v) \leq \\ &\underbrace{d^{H^+_{G_L}}(v) + m(v, L, G) + w(v)}_{r^w(v,L,G)}. \end{aligned}$$

Finally, if we take the maximum of the left side and right side of the inequality above we can conclude that

$$\bar{\Delta}^{H^+_{G_{L'}}} \leq r^w(L, G)$$

where $L' \in \Pi(G)$ stands for the linear ordering which is obtained, if the game ends, i.e., the order the vertices were marked. □

The Game Chromatic Number of Weighted Planar Graphs

By the proposition 3.6.2 the weighted marking game number is bounded by the weighted rank. The task of determining the game chromatic number of a weighted graph G or a class of weighted graphs is reduced to the task of determining the weighted rank. In the following we will use this result to give an upper bound for the game chromatic number of weighted planar graphs.

Corollary 3.6.3. *If G is a weighted planar graph, then*

$$col^w_g(G) \leq \min_{L \in \Pi(G)} \max_{v \in V(G)} \left\{ w(v) + 17 - \left(d^+_{G_L}(v) - d^{H^+_{G_L}}(v) \right) \right\}.$$

Proof. Let G be a weighted planar graph. Kierstead found out in [11] that at any time for an unchosen vertex $v \in V(G)$ we have:

$$d^+_{G_L}(v) + m(v, L, G) \leq 17.$$

Using this result, we are able to conclude for the vertex $v \in V(G)$

$$r^w(v, L, G) = w(v) + d^{H^+_{G_L}}(v) + m(v, L, G) + d^+_{G_L}(v) - d^+_{G_L}(v) \leq$$

$$w(v) + 17 - \left(d^+_{G_L}(v) - d^{H^+_{G_L}}(v)\right).$$

Hence,

$$r^w(L, G) \leq \max_{v \in V(G)} \left\{w(v) + 17 - \left(d^+_{G_L}(v) - d^{H^+_{G_L}}(v)\right)\right\}$$

and

$$r^w(G) \leq \min_{L \in \Pi(G)} \max_{v \in V(G)} \left\{w(v) + 17 - \left(d^+_{G_L}(v) - d^{H^+_{G_L}}(v)\right)\right\}.$$

Thus, by proposition 3.6.2 the proof is completed. $\square$

Chapter 4

The General Asymmetric Game on Graphs

In this chapter we will restrict our attention to the *asymmetric game on graphs*, also called the (a, b)*-coloring game*, which was introduced by Kierstead in [17]. It differs from Bodlaender's two-person game in that way that on each turn each player has to color several vertices in a row instead of one vertex. Alice colors a and Bob b vertices for $a, b \in \mathbb{N}$, $a, b \geq 1$ in a row. Note that the respective player does not have to complete his turn as soon as either every vertex is colored or a feasible coloring of the graph is not possible (see page 9 for a detailed definition).

We intend to generalize Kierstead's asymmetric game as follows: Our game is based on the idea that the number of the vertices to color varies each time the players take turns, provided that they are allowed to make several moves in a row as in the (a, b)-coloring game. Thus, we consider an m-tuple $(x_1, ..., x_m)$ for a player, such that x_i vertices are being colored in the ith move for $i \in \{1, ..., m\}$. This new consideration provides a more natural characterization of the regular asymmetric game since the number of the moves is variable. We will discuss the new game, called *general asymmetric game on graphs*, for the class of cycles, complete multipartite graphs and forests, while determining the *general asymmetric game chromatic number* $\gamma_g(G; a, b)$ for some relevant distributions of the m-tuples. For the purpose of bounding the new parameter for the class of forests, we will extend the marking game on graphs (see page 8) to the *general asymmetric marking game on graphs* and define the *general asymmetric marking game number*, denoted by $col_g^{ga}(G; a, b)$. In particular, we will prove that $\gamma_g(G; a, b) \leq col_g^{ga}(G; a, b)$ holds for a graph G.

The general asymmetric coloring game on graphs

Definition 4.0.4. Let C be a set of given colors and let $a = (a_1, a_2, ..., a_m)$ and $b = (b_1, b_2, ..., b_m)$ with $a_i, b_i \in \mathbb{N}$, $a_i, b_i \geq 1$ for $i \in \{1, ..., m\}$. Further, let $\sum_{i=1}^{m} a_i + \sum_{i=1}^{m} b_i \geq n$. Two players Alice and Bob take turns assigning colors from C to vertices of G from the shrinking set W of uncolored vertices until $W = \emptyset$ such that adjacent vertices get distinct colors. The tth move of Alice and Bob consists of coloring a_t and b_t vertices, respectively. Alice starts the game and wins if there is no uncolored vertex left. Otherwise, Bob wins.

For $a = (a_1, a_2, ..., a_m)$ and $b = (b_1, b_2, ..., b_m)$ the *general asymmetric game chromatic number* $\gamma_g(G; a, b)$ of a graph $G = (V, E)$ is the least cardinality of C such that there exists a winning strategy for Alice when the general asymmetric coloring game is played on G with the given set C of colors.

Obviously for the ordinary asymmetric game chromatic number it holds that $\gamma(G; a, b) = \gamma_g(G; a, b)$, whenever $x_i = x_j$ for $x \in \{a, b\}$ and $i, j \in \{1, ..., m\}$. Moreover, $\gamma(G) = \gamma_g(G; a, b)$, if $a_i = b_i = 1$ for all $i \in \{1, ..., m\}$.

4.1 The General Asymmetric Game Chromatic Number of Cycles

We denote a cycle with n vertices by C_n where $V(C_n) = \{x_1, ..., x_n\}$. We are interested in finding out $\gamma_g(C_n; a, b)$, where $a = (a_1, ..., a_m)$, $b = (b_1, ..., b_m)$ for n even and odd. For this purpose let us first determine the regular asymmetric game chromatic number $\gamma(C_n; a, b)$, where the number of the moves of the players does not vary each time they take turns. Afterwards, we will consider a cycle with a *chord* which is an edge between two vertices x_i and x_j for $(x_i, x_j) \notin E(C_n)$, $i, j \in \{1, ..., n\}$; in particular, we will determine $\gamma(G; 1, 2)$, $\gamma(G; 1, b)$ for all $b > 2$ and $\gamma(G; a, b)$ for all integers b and $a > 1$.

The asymmetric game chromatic number of cycles

Lemma 4.1.1. *Let $C_n = (V, E)$ be a cycle with n even and $n \geq 4$. Then*

(i) $\gamma(C_n; a, b) = 2$ *for* $a \geq \lceil \frac{n}{3} \rceil$ *and* $b \in \mathbb{N}$ *and*

(ii) $\gamma(C_n; a, b) = 3$ *for* $a < \lceil \frac{n}{3} \rceil$ *and* $b \in \mathbb{N}$.

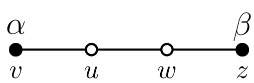
Proof. (i) Assume 2 colors α and β are given. Further, let $a = \lceil \frac{n}{3} \rceil$. We will prove a winning strategy for Alice, denoted by σ, where she can fix the coloring of the graph after her first turn.

Strategy σ : Alice colors in the clockwise direction every third vertex with α and β alternately as follows:

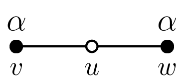
- If $n \bmod 3 = \{0, 1\}$, then she colors vertices x_{1+6j} with α and x_{4+6j} with β for all $j \in \{0, ..., \lceil \frac{n}{6} \rceil - 1\}$.
- If $n \bmod 3 = 2$, then she colors vertices x_{1+6j} with α for all $j \in \{0, ..., \lceil \frac{n}{6} \rceil - 1\}$ and x_{4+6i} with β for all $i \in \{0, ..., \lfloor \frac{n}{6} \rfloor - 1\}$.

Obviously, if Alice plays strategy σ, at least every third vertex is colored with α and β alternately. Thus, the coloring of the entire graph is fixed, if every uncolored vertex u with $\{v, w\} = N(u)$ satisfies either property $\mathcal{P}_1$ or property $\mathcal{P}_2$, where

$\mathcal{P}_1$: Let $z \in N(w)$. v and z are colored with different colors and w is uncolored.



$\mathcal{P}_2$: v and w are colored with the same color.



Since every uncolored vertex x_i for $i \in \{1, ..., n-2\}$ satisfies $\mathcal{P}_1$ or $\mathcal{P}_2$ we are left with the task of proving that either x_n or x_{n-2} has the color β or x_{n-1}

the color α.

Consider the case $n \bmod 3 = 0$. Then we can conclude that $a = \frac{n}{3}$ and that a is even. Clearly, Alice colors until she reaches x_{n-2} where $\frac{a}{2}$ vertices have been colored with α and β, respectively; in particular x_1 has been colored with α and x_{n-2} with β. Obviously, this fixes the coloring and Alice wins the game.

Consider the case $n \bmod 3 = 1$. Then $n = 3l + 1$ for $l \in \mathbb{N}$. We can conclude that

$$a = \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{3l+1}{3} \right\rceil = l + \left\lceil \frac{1}{3} \right\rceil = l + 1. \tag{4.1}$$

Since n is even, by the assumption $n = 3l + 1$, $3l$ is odd, which implies that l is odd. Thus, according to 4.1 a is even. Hence, $\frac{a}{2}$ vertices are colored with α and β, respectively. Thus, x_1 is colored by Alice with α and x_n with β since $n \bmod 3 = 1$. This fixes the coloring and hence Alice wins the game.

Consider the case $n \bmod 3 = 2$. Then $n = 3l + 2$ for $l \in \mathbb{N}$. Thus,

$$a = \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{3l+2}{3} \right\rceil = l + \left\lceil \frac{2}{3} \right\rceil = l + 1. \tag{4.2}$$

Since n is even, by the assumption $n = 3l + 2$, $3l$ is even, too, which implies that l is even. Thus, by 4.2 a is odd. Hence, $\lceil \frac{a}{2} \rceil$ vertices are colored with α and $\lfloor \frac{a}{2} \rfloor$ vertices are colored with β. Thus, x_1 is colored by Alice with α and $x_n - 1$ with α again since $n \bmod 3 = 2$. This fixes the coloring and hence Alice wins the game.

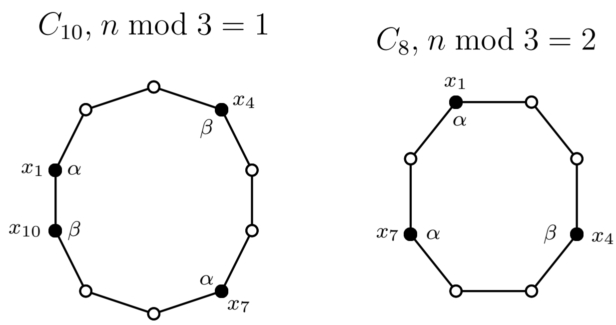
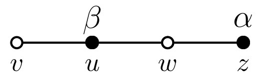


FIGURE: Alice achieves to fix the coloring of C_{10} and C_8 after the first turn.

(ii) Let n be even and assume that two colors α and β are given. Since $a < \lceil \frac{n}{3} \rceil$ we can conclude from the pigeon hole principle that after Alice's first turn there exists at least one sequence of vertices $\{v, u, w, z\}$ with z colored and $\{v, u, w\}$ uncolored. Without loss of generality assume that z is colored with α and it is Bob's first turn. Then he is able to attack w by coloring u with β which is valid because v is uncolored yet. Then Bob wins the game.



□

Remark 4.1.2. Let $C_n = (V, E)$ be a cycle with n odd. Then $\gamma(C_n; a, b) = 3$ for $a, b \in \mathbb{N}$.

Proof. The proof is based on the following observation:

$$3 = \chi(C_n) \leq \gamma(C_n; a, b) \leq \Delta(C_n) + 1 = 3.$$

□

Since the degree of each vertex on a cycle is 2 the asymmetric game chromatic number of it is at most 3, independent of the integers a and b. A more interesting result can be obtained if we consider a cycle which contains an edge that joins two vertices on the cycle but is not itself an edge of it. In this way we obtain a cycle with two vertices with degree 3, that can be attacked more than twice. Such an edge is called *chord* and is defined as follows:

Definition 4.1.3. Let $G = (V, E)$ be a graph and $C_n \in G$ a cycle on n vertices. We call an edge $e \in E(G)$ a *chord* if it joins two vertices of C_n but $e \notin E(C_n)$.

In the following we are going to consider a cycle on n vertices with a chord and analyze the asymmetric game chromatic number for all integers a and b. From now on, we will denote the two cycles that are created by a chord by C^1 and C^2. Without loss of generality assume that $|V(C^1)| \leq |V(C^2)|$.

Lemma 4.1.4. *Let $G = (V, E)$ be a graph that consists of a cycle C_n on n vertices with $n > 3$ and a chord e on C_n.*

(i) Let $n \leq 5$. Then $\gamma(G; 1, 2) = 4$.

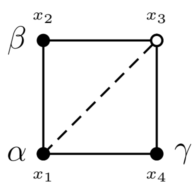
(ii) Let $n = 6$.

- *If $|V(C^1)| = 3$, then $\gamma(G;1,2) = 3$.*
- *If $|V(C^1)| = |V(C^2)| = 4$, then $\gamma(G;1,2) = 4$.*

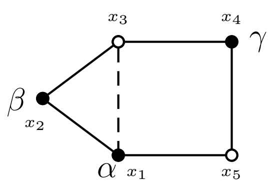
(iii) Let $n \geq 7$. Then $\gamma(G;1,2) \in \{3,4\}$.

Proof. Consider $V(G) = \{x_1, ..., x_n\}$, such that $e = (x_1, x_m)$ for $3 \leq m \leq n-1$. Assume 3 colors α, β and γ are given. Since the degree of x_1 and x_m is 3, the worst case occurs, if Bob achieves to attack one of them for three times.

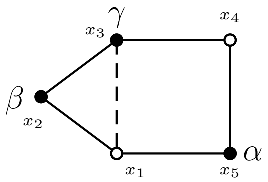
(i) Let $n = 4$, then $m = 3$ and $e = (x_1, x_3)$ is a chord. Because of the pigeon hole principle, independent of which vertex Alice colors first, she attacks simultaneously at least one vertex of the chord. Assume she colors a neighbor of vertex x_3 with α. Since $b = 2$ and x_3 has degree 3, Bob replies by coloring the remaining uncolored two neighbors of x_3 by assigning them the colors β and γ, respectively. Since there is no available color left for x_3, Bob wins the game.



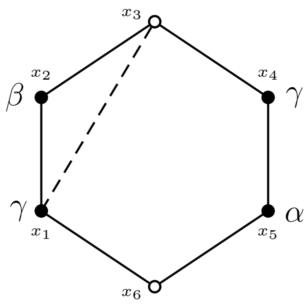
Assume $n = 5$, Then $m \in \{3,4\}$. Without loss of generality assume that $m = 3$, such that $e = (x_1, x_3)$ is a chord. In the same manner as above Alice attacks at least one vertex from (x_1, x_3), independent of which vertex she colors first: By symmetry, Alice has two possibilities for coloring the first vertex, namely to color x_i either for $i \in \{1,3\}$ or for $i \in \{2,4,5\}$. Suppose she colors vertex x_1 with α. Then Bob wins the game by assigning vertices x_2 and x_4 the colors β and γ, respectively, because there is no available color left for x_3.



Suppose Alice colors vertex x_i for $i \in \{2, 4, 5\}$ with α. Then because of the assumption $n = 5$ at least one vertex from e is adjacent to x_i. Thus, Bob achieves two attack either x_1 or x_3 twice and hence 4 colors are needed to guarantee Alice's victory.



(ii) Let $n = 6$ and $|V(C^1)| = 3$. Then $m = 3$ and (x_1, x_3) is a chord. Since $|V(C^2)| = 5$, there exists $y \in V(C^2)$ with $d_{(x_1,y)} = d_{(x_3,y)} = 2$. Obviously, $y = x_5$. Alice's winning strategy is to color vertex x_5 with α. Since neither x_1 nor x_3 are adjacent to x_5, none of the vertices on the chord has colored neighbors.



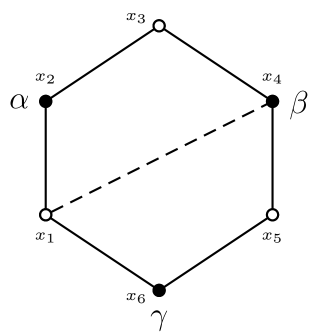
If Bob colors at least one vertex frm $\{x_1, x_3\}$, then the coloring of all vertices with three vertices is trivial. Thus consider the following two cases:

(i) Bob colors x_2 and a vertex from $\{x_4, x_6\}$. Without loss of generality assume that he decides to attack x_3 by coloring x_2 and x_4 with β and γ, respectively. Then Alice wins the game by coloring x_1 with γ, which is possible since $(x_1, x_4) \notin E(G)$. The coloring of the remaining uncolored vertices is fixed now. Otherwise, if Bob colored both with the same color, then Alice would color x_1 with another color than β.

(ii) Bob colors x_4 and x_6. If both are colored with the same color, say β, then Alice would color x_2 with β. Otherwise, assume x_4 is colored with β and x_6 with γ. Then Alice would color x_1 with β or x_3 with γ.

Assume 2 colors α and β are given. Since n is sufficiently big, Bob can attack a vertex on C_n once when he takes his first turn. Thus, we have $\gamma_g(G; 1, 2) = 3$.

Let $|V(C^1)| = |V(C^2)| = 4$. Then $m = 4$, such that $e = (x_1, x_4)$ is a chord. Independent of which vertex Alice decides to color first, she attacks either x_1 or x_4 because there exists no vertex $y \in V(G)$ with $d_{(y,x_1)} > 1$ and $d(y, x_4) > 1$. Assume x_1 has a neighbor which is colored with α. Since x_1 is contained in the chord, it has degree 3, such that by the assumption $b = 2$ Bob is able to attack x_1 twice in his next turn by coloring the remaining vertices with colors β and γ, respectively. Thus, a feasible coloring of x_1 is not possible anymore and Bob wins the game. Thus, we have $\gamma_g(G; 1, 2) = 4$.



(iii) Let $n \geq 7$.

Case 1: Assume $|V(C^1)|, |V(C^2)| \geq 4$ and let $e = (x_1, x_j)$ be a cord. In her first turn Alice won't color x_1, x_j or a neighbor of x_1, x_j, since this gives Bob the opportunity to attack either x_1 or x_j twice and hence to produce his best case. Thus, Alice colors a vertex different from x_1, x_j and different from a neighbor of x_1 and x_j. Assume she colors vertex x_k for $k \notin \{1, 2, j-1, j, j+1, n\}$ with α.

- If $k > j + 1$, then Bob's winning strategy is to color x_{j+1} and x_n with β.
- If $k < j - 1$, the Bob's winning strategy is to color x_2 and x_{j-1} with β.

x_1 and x_j have now a colored neighbor with color β which makes it for Alice impossible to "save" both vertices with one move. By coloring x_1, she attacks x_j and vice versa. If she colors a neighbor of x_1 or x_j, Bob attacks x_j or x_1 twice in his next move. Hence, in Bob's second turn he achieves to attack either x_1 or x_j twice such that $\gamma_g(G; 1, 2) = 4$.

Case 2: Assume $|V(C^1)| = 3$ and let $e = (x_1, x_3)$ be the cord. Again Alice won't color x_1, x_3 or a neighbor of x_1, x_3 in her first move. Contrary to case 1, if Bob colors a neighbor of x_1 and a neighbor of x_3, Alice can avoid the worst case because x_2 is adjacent to x_1 and x_3: In particular if Bob colors x_n (which is adjacent to x_1) and x_4 (which is adjacent to x_3) with the same color, say α, Alice colors x_2 also with α and hence 3 colors suffice to color the whole graph. If Bob colors x_n and x_4 with different colors, Alice colors x_3 with the same color as x_n is colored and again 3 colors suffice for Alice to win the game. If Bob colors x_2 and another neighbor of x_1 or x_3, Alice again easily avoids the worst case scenario. Thus, Bob also decides to color vertices different from x_1, x_3 and different from a neighbor of x_1 and x_3.

- If $n - 5 \bmod 3 = 1$, then Bob is the first who colors two vertices of either x_1, x_3 or a neighbor of x_1 and x_3. As showed above 3 colors suffice then for Alice to win the game.

- If $n - 5 \bmod 3 = 0$, then Alice is the first who colors either x_1, x_3 or a neighbor of x_1 and x_3. Thus, Bob produces now his best case and attacks either x_1 or x_3 twice.

- If $n-5 \bmod 3 = 2$ for $n > 7$, then Bob is the first who colors either x_1, x_3 or a neighbor of x_1 and x_3. If he colors either x_1, x_3, x_4 or x_n, Alice obviously can avoid the worst case. Hence, assume that he colors x_2 with the same color as x_5 and x_{n-1} (since $b = 2$ Bob can easily achieve that x_5 and x_{n-1} get the same color). Without loss of generality assume that he colors x_2 with α. If Alice colors in her turn x_1 or x_3, then Bob is able to attack, respectively, x_3 or x_1 twice. If Alice colors x_4 or x_n, she has to use a color different from α, which gives Bob again the opportunity to attack x_1 or x_3 twice.

  Let $n = 7$. It is easily seen that three colors are suficient for coloring the graph, if Alice colors x_5 or x_6 in her first turn. The details are left to the reader. $\square$

Lemma 4.1.5. *Let $G = (V, E)$ be a graph that consists of a cycle C_n on n vertices with $n > 3$ and a chord e on C_n. Then for all positive integers $b > 2$ it holds $\gamma(G; 1, b) = 4$.*

Proof. Since G is a cycle with a chord, there exist two vertices x and y with degree 3. Assume 3 colors are given. Independent of which strategy Alice applies after her first turn either x or y will stay uncolored by the assumption $a = 1$. Without loss of generality assume that x is uncolored. Then by the assumption $b \geq 3$ after Bob's first turn the neighbors of x will be colored with distinct colors. Thus, a feasible coloring of G won't be possible anymore since there won't be an available color left for x. □

Lemma 4.1.6. *Let $G = (V, E)$ be a graph that consists of a cycle C_n on n vertices with $n > 3$ and a chord e on C_n. Then for all positive integers $a > 1$ and b, $\gamma(G; a, b) \leq 3$.*

Proof. Let $e = (x, y)$ be the chord. Assume 3 colors are given. Then Alice's winning strategy is to color x and y in her first move, which is possible by the assumption $a \geq 2$. Since any other vertex on C_n has degree 2 and can be attacked at most twice, 3 colors suffice to ensure Alice's victory. □

General Asymmetric Game Chromatic Number of Cycles

Corollary 4.1.7. *Let $C_n = (V, E)$ be a cycle with $n \geq 4$ even and $a = (a_1, ..., a_k), b = (b_1, ..., b_k)$ with $a_i, b_i \in \mathbb{N}$ for $1 \leq i \leq k$. Then*

(i) $\gamma_g(C_n; a, b) = 2$ *for* $a_1 \geq \lceil \frac{n}{3} \rceil$ *and*

(ii) $\gamma_g(C_n; a, b) = 3$ *for* $a_1 < \lceil \frac{n}{3} \rceil$.

Proof. (i) Assume two colors are given. Then by the assumption $a_1 \geq \lceil \frac{n}{3} \rceil$ we can refer to the proposition 4.1.1 (i), where we proved a winning strategy for Alice with two colors such that she wins the game after her first turn for $a \geq \lceil \frac{n}{3} \rceil$.

(ii) Assume two colors are given and let $b_1 = 1$. Then by the assumption $a_1 < \lceil \frac{n}{3} \rceil$ we can refer to the proposition 4.1.1 (ii), where Bob wins the game after his first turn. □

Corollary 4.1.8. *Let $C_n = (V, E)$ be a cycle with n odd and $a = (a_1, ..., a_k), b = (b_1, ..., b_k)$ with $a_i, b_i \in \mathbb{N}$ for $1 \leq i \leq k$. Then $\gamma_g(C_n; a, b) = 3$.*

Proof. The proof runs as in 4.1.2. □

4.2 The General Asymmetric Game Chromatic Number of Complete Multipartite Graphs

For the purpose of determining the general asymmetric game chromatic number of complete multipartite graphs, in analogy to the previous section, first we will restrict our attention to the regular asymmetric game for such graphs.

The Asymmetric Game Chromatic Number of Complete Multipartite Graphs

We determine the asymmetric game chromatic number of the class of complete multipartite graphs for all values $a, b \in \mathbb{N}$. For the choice of the vertices to color, we refer to the winning strategy of Alice that we worked out in proposition 2.3.2 for the circular two-person game on complete multipartite graphs.

Alice colors every time in an independent set which contains only uncolored vertices, if possible. Once every independent set contains at least one colored vertex, the coloring is fixed. From this point she colors vertices at random. Contrary, Bob's optimal strategy is to color vertices in independent sets that contain colored vertices using every time new colors, if possible. Otherwise, he colors the vertices at random.

According to the optimal strategies of both players, we prove further how many colors are required to guarantee Alice's victory in the asymmetric game. For the rest of this section we will denote a complete multipartite graph with $N_1, ..., N_m$ independent sets by $K_{n_1,...,n_m}$ where $|N_i| = n_i$ for all $i \in \{1, ..., m\}$. For the remainder of this section we assume that $n_1 \leq n_2 \leq ... \leq n_m$.

Lemma 4.2.1. *Let $K_{n_1,...,n_m} = (V, E)$ be a complete multipartite graph with $a \geq m$ and $a, b \in \mathbb{N}$. Then $\gamma(K_{n_1,...,n_m}; a, b) = m$.*

Proof. By the assumption $a \geq m$, after Alice's first move, each independent set contains at least one colored vertex. Thus, m colors are sufficient for coloring the graph. □

We are left with the task of determining $\gamma(K_{n_1,...,n_m}; a, b)$ for $a < m$. For this case we need a little refinement of Alice's strategy mentioned above because she is not able to fix the coloring of each independent set after her first turn. In the following we will give Alice's winning strategy and work out Bob's worst case strategy. Further, let C be the set of all colored vertices during the game. Each time a vertex is colored Alice updates C.

Alice's strategy: Let $L = \{N_1, N_2, ..., N_m\}$. Initially, Alice colors one vertex in each independent set from $N_1, ..., N_a$. In her ith turn she searches along L a uncolored independent sets with the least index and colors one vertex in each set. If the amount of all uncolored independent sets is less than a, then the coloring is fixed as soon as each independent set contains at least one colored vertex.

Bob's strategy: Each time he takes turn he colors vertices in independent sets that contain colored vertices, if possible, while using every time new colors. Otherwise, he proceeds as follows: Let $j \leq m$ be the greatest integer such that N_j does not contain colored vertices. Then he colors all vertices in N_j with new colors until every vertex is colored. Then he jumps to N_{j-1} and goes on until $N_i \cap C \neq \emptyset$ for all $i \in \{1, ..., m\}$.

The strategies above are the best possible for Alice and Bob. This is easy to see by the following consideration. By the assumption $n_1 \leq n_2 \leq ... \leq n_m$ it makes sense, if Alice colors in $N_1, ..., N_a$ in her first turn so that the amount of the remaining uncolored vertices is minimal and hence the amount of attacks made by Bob. Consider for example the case $b > \big(\sum_{i=1}^{a} n_i\big) - a$ and $b \leq \big(\sum_{j=m-a+1}^{m} n_j\big) - a$.

- If Alice colors in the sets $N_1, ..., N_a$ first, then Bob will be forced to jump at least once in a new independent set. This implies that the coloring of at least one independent set is fixed by Bob. In particular, he would jump to N_m so that he maximizes the amount of attacks he makes and minimizes the amount of jumps into uncolored independent sets. If N_m is completely colored, then he jumps to N_{m-1} and so on until he reaches N_a.

- If Alice colors in $N_{m-a+1}, ..., N_m$ first, then Bob will be able to make b attacks without coloring in an uncolored set.

Lemma 4.2.2. *Let $K_{n_1,...,n_m} = (V, E)$ be a complete multipartite graph with $a < m$ and $a + b \geq \sum_{i=1}^{m} n_i - n_{a+1} + 1$ for $a, b \in \mathbb{N}$. Further, let $n_1 \leq n_2 \leq ... \leq n_m$. Then $\gamma(K_{n_1,...,n_m}; a, b) = \sum_{i=1}^{m} n_i - n_{a+1} + 1$.*

Proof. Alice's winning strategy is to color a vertices in $N_1, ..., N_a$, in her first turn. Since $a + b \geq \sum_{i=1}^{m} n_i - n_{a+1} + 1$, by Bob's worst case strategy, he first colors the remaining uncolored vertices in $N_1, ..., N_a$ and goes on with $N_m, N_{m-1}, ..., N_{a+2}$. As soon as he jumps into N_{a+1} and colors one vertex, the coloring of the graph is fixed. Thus, we can conclude that Alice uses a and Bob $(n_1 + ... + n_a) - a + (n_m + ... + n_{a+2}) + 1$ colors, respectively. Thus,

$$\gamma(K_{n_1,...,n_m}; a, b) \leq \sum_{i=1}^{m} n_i - n_{a+1} + 1.$$

Obviously, if less than $\sum_{i=1}^{m} n_i - n_{a+1} + 1$ colors are given, a feasible coloring of at least N_{a+1} won't be possible. $\square$

A more general case can be obtained if we assume that the game does not end after the first round which consists of one move of Alice and Bob, respectively. Assume we have a complete multipartite graph $K_{n_1,...,n_m}$ with $a < m$ and

$$a + b < 1 + \sum_{i=1}^{a} n_i + \sum_{j=a+2}^{m} n_j \tag{4.3}$$

for $a, b \in \mathbb{N}$. Assume Alice and Bob apply their optimal strategies, respectively. Since b can be arbitrarily large under the condition (4.3), it can happen that Bob will be forced to jump into an uncolored independent set. Next we determine the amount of jumps Bob will make during the game. Obviously, if

$$\Big(\sum_{i=1}^{a} n_i\Big) - a \geq b,$$

Bob will not jump after his first turn and hence also during the whole game since $n_1 \leq n_2 \leq ... \leq n_m$.

Assume $\left(\sum_{i=1}^{a} n_i\right) - a < b$.

First round: Bob will be forced to jump at least once. Precisely, let

$$\bar{x}_1 = \max\left\{a+1 \leq x \leq m \;\middle|\; \left(\sum_{i=1}^{a} n_i\right) - a + \sum_{j=x}^{m} n_j \geq b\right\},$$

then $J_1 = m - \bar{x}_1 + 1$ is the amount of jumps done by Bob after his first turn since he jumped to $N_m, ..., N_{\bar{x}_1}$ and $r_1 = \left(\sum_{i=1}^{a} n_i\right) - a + \left(\sum_{j=\bar{x}_1}^{m} n_j\right) - b$ are the remaining uncolored vertices of $N_{\bar{x}_1}$.

Second round: Assume it is Alice's turn. If $(a + J_1) + a < m$, then after Alice's turn there will exist uncolored independent sets. Due to her strategy Alice colors another a vertices such that $\left(\sum_{i=a+1}^{2a} n_i\right) - a$ uncolored vertices are left in $N_{a+1}, ..., N_{2a}$.

If $\left(\sum_{i=a+1}^{2a} n_i\right) - a + r_1 < b$, Bob will be forced to jump to an uncolored independent set again. Moreover, if $(2a + b) + b < 1 + \sum_{i=1}^{2a} n_i + \sum_{j=2a+2}^{m} n_j$, then after Bob's turn there still exist uncolored independent sets. Precisely let

$$\bar{x}_2 = \max\left\{2a+1 \leq x \leq \bar{x}_1 - 1 \;\middle|\; \left(\sum_{i=a+1}^{2a} n_i\right) - a + r_1 + \sum_{j=x}^{\bar{x}_1 - 1} n_j \geq b\right\},$$

then $J_2 = \bar{x}_1 - \bar{x}_2$ is the amount of jumps done by Bob in his second turn since he jumped to $N_{\bar{x}_1 - 1}, ..., N_{\bar{x}_2}$ and $r_2 = \left(\sum_{i=a+1}^{2a} n_i\right) - a + \left(\sum_{j=\bar{x}_2}^{\bar{x}_1 - 1} n_j\right) + r_1 - b$ are the remaining uncolored vertices of $N_{\bar{x}_2}$.

In this manner we go on and can conclude for the $l+1$-th round the following:

$(l+1)$th round: Assume it is Alice's turn. If $(la + \sum_{i=1}^{l} J_i) + a < m$, then after Alice's turn there will exist uncolored independent sets.

Assume it is Bob's turn. As long as $\left(\sum_{i=la+1}^{(l+1)a} n_i\right) - a + r_l < b$ Bob will be forced to jump in at least one uncolored independent set and if $\big((l+1)a + lb\big) +$

$b < 1 + \sum_{i=1}^{(l+1)a} n_i + \sum_{j=(l+1)a+2}^{m} n_j$, then after Bob's turn there will exist uncolored independent sets.

We calculate

$$\bar{x}_{l+1} = \max\Big\{(l+1)a + 1 \leq x \leq \bar{x}_l - 1 \;\Big|\; \Big(\sum_{i=la+1}^{(l+1)a} n_i\Big) - a + r_l + \sum_{j=x}^{\bar{x}_l - 1} n_j \geq b\Big\},$$

and conclude $J_{l+1} = \bar{x}_l - \bar{x}_{l+1}$ is the number of jumps done by Bob in his $l+1$th turn with $r_{l+1} = \Big(\sum_{i=la+1}^{(l+1)a} n_i\Big) - a + \Big(\sum_{j=\bar{x}_{l+1}}^{\bar{x}_l - 1} n_j\Big) + r_l - b$.

It remains to consider the case that one player colors in the last uncolored independent set, such that the coloring from this point on is fixed. Assume that this happens in the l_0th round and that Alice is the player, who colors in the last uncolored independent set. We can conclude that before Alice's move we have $\big((l_0-1)a + \sum_{i=1}^{l_0-1} J_i\big) + a \geq m$ and that $\sum_{i=1}^{l_0-1} J_i$ is the number of jumps Bob made during the game. Next assume that Bob is the player who colors in the last independent set. Before Bob's move we have $\Big(\sum_{i=(l_0-1)a+1}^{l_0 a} n_i\Big) - a + r_{l_0-1} < b$ (since he jumps at least once) and $\big(l_0 a + (l_0-1)b\big) + b \geq 1 + \sum_{i=1}^{l_0 a} n_i + \sum_{j=l_0 a+2}^{m} n_j$ because he colors first in the last independent set. The amount of jumps done by Bob during the game equals $\sum_{i=1}^{l_0-1} J_i + \big(m - (l_0 a + \sum_{i=1}^{l_0-1} J_i)\big)$, where clearly $m - (l_0 a + \sum_{i=1}^{l_0-1} J_i)$ is the number of jumps done by Bob in his l_0th move.

For the rest of this section we will denote by h the number of jumps done by Bob during the game. Moreover, if Alice fixes the coloring let $r := \min\Big\{k \in \mathbb{N} \;\Big|\; \sum_{i=1}^{k} a \geq m-h\Big\}$ be the move in which this happens. If Bob fixes the coloring then let $t := \min\Big\{1 \leq k \leq m \;\Big|\; \sum_{i=1}^{k}(a+b) \geq |V(K_{n_1,\dots,n_m})| - n_{ka+1} + 1\Big\}$ be the move in which this happens.

Lemma 4.2.3. *Let $K_{n_1,\dots,n_m} = (V, E)$ be a complete multipartite graph with $a < m$*

and $a + b < 1 + \sum_{i=1}^{a} n_i + \sum_{j=a+2}^{m} n_j$ *for* $a, b \in \mathbb{N}$.

- *If Alice fixes the coloring, then it holds* $\gamma(K_{n_1,...,n_m}; a, b) = m - h + (r-1)b$.
- *If Bob fixes the coloring, then it holds* $\gamma(K_{n_1,...,n_m}; a, b) = \sum_{i=1}^{ta} n_i + \sum_{j=ta+2}^{m} n_j + 1$.

Proof. Assume as above that the players apply their optimal strategies. Once every independent set contains at least one colored vertex the coloring is fixed. Clearly, either Alice or Bob will fix the coloring.

- Assume Alice fixes the coloring. Clearly, she makes r moves, while Bob makes $r-1$ moves. According to her optimal strategy Alice jumps every time in an uncolored independent set and hence she uses $ra - (ra - (m - h))$ colors. Bob uses $(r-1)b$ colors. Hence, $ra - (ra - (m-h)) + (r-1)b$ colors are required to guarantee Alice's victory.
- Assume Bob fixes the coloring. Alice uses ta colors while Bob uses $\left(\sum_{i=1}^{ta} n_i\right) - ta + \left(\sum_{j=ta+2}^{m} n_j\right) + 1$ colors. Hence, $\left(\sum_{i=1}^{ta} n_i\right) + \left(\sum_{j=ta+2}^{m} n_j\right) + 1$ colors are needed to guarantee Alice's victory.

$\square$

The General Asymmetric Game Chromatic Number of Complete Multipartite Graphs

In this section we will extend lemma 4.2.3 for the case of general asymmetric games. Since the number of the vertices to color varies each time the players take turns, we need to calculate the amount of jumps into uncolored independent sets Bob will make during the game. For this purpose we refer to the method introduced in the section above so that we just need to apply the general case. Again we denote a complete multipartite graph with $N_1, ..., N_m$ independent sets by $K_{n_1,...,n_m}$ where $|N_1| = n_1 \leq |N_2| = n_2 \leq ... \leq |N_m| = n_m$. Further, let $a = (a_1, ..., a_k)$ and $b = (b_1, ..., b_k)$ be the tuples for the general asymmetric game for Alice and Bob, respectively, where x_i vertices are being colored in the ith round for $x \in \{a, b\}$ and $i \in \{1, ..., k\}$. Moreover, we define $A_l := \sum_{i=1}^{l} a_i$ and $B_l := \sum_{i=1}^{l} b_i$.

(l + 1)th round: Suppose it is Alice's turn and that $(A_l + \sum_{i=1}^{l} J_i) + a_{l+1} < m$. Then after Alice's move there still exists at least one uncolored independent set.

Suppose it is Bob's turn and assume that $\Big(\sum_{i=A_l+1}^{A_{l+1}} n_i\Big) - a_{l+1} + r_l < b_{l+1}$ and that $(A_{l+1} + B_l) + b_{l+1} < 1 + \sum_{i=1}^{A_{l+1}} n_i + \sum_{j=A_{l+1}+2}^{m} n_j$. Then after Bob's move there still exists at least one uncolored independent set and we calculate

$$\bar{x}_{l+1} = \max_x \left\{ A_l + 1 \le x \le \bar{x}_l - 1 \;\middle|\; \Big(\sum_{i=A_l+1}^{A_{l+1}} n_i\Big) - a_{l+1} + r_l + \sum_{j=x}^{\bar{x}_l - 1} n_j \ge b_{l+1} \right\},$$

and conclude $J_{l+1} = \bar{x}_l - \bar{x}_{l+1}$ being the number of jumps made by Bob in his $l + 1$ turn with

$$r_{l+1} = \Big(\sum_{i=A_l+1}^{A_{l+1}} n_i\Big) - a_{l+1} + \Big(\sum_{j=\bar{x}_{l+1}}^{\bar{x}_l - 1} n_j\Big) + r_l - b_{l+1}$$

being the remaining uncolored vertices of the independent set which contains at least one colored vertex.

Again we consider further the case that one player fixes the coloring. Assume this happens in round l_0 and that Alice fixes the coloring. Before Alice's move it holds then $(A_{l_0-1} + \sum_{i=1}^{l_0-1} J_i) + a_{l_0} \ge m$ and we have $h_g = \sum_{i=1}^{l_0-1} J_i$, where h_g stands for the number of jumps done by Bob during the game. Next assume that Bob fixes the coloring. Then before Bob's turn we have

$$\Big(\sum_{i=A_{l_0-1}+1}^{A_{l_0}} n_i\Big) - a_{l_0} + r_{l_0-1} < b_{l_0},$$

because by the assumption he must jump to an uncolored independent set at least once and

$$(A_{l_0} + B_{l_0-1}) + b_{l_0} \ge 1 + \sum_{i=1}^{A_{l_0}} n_i + \sum_{j=A_{l_0}+2}^{m} n_j,$$

since he fixes the coloring. The number of jumps done by Bob during the game equals $h_g = \sum_{i=1}^{l_0-1} J_i + \Big(m - (A_{l_0} + \sum_{i=1}^{l_0-1} J_i)\Big)$, where $J_{l_0} = m - (A_{l_0} + \sum_{i=1}^{l_0-1} J_i)$.

We introduce further two parameters. If Alice fixes the coloring let $r_g := \min\Big\{l \in \mathbb{N} \;\Big|\; \sum_{i=1}^{l} a_i \geq m - h_g\Big\}$ be the move in which this happens. If Bob fixes the coloring then let $t_g := \min\Big\{1 \leq l \leq m \;\Big|\; \sum_{i=1}^{l}(a_i + b_i) \geq |V(K_{n_1,\dots,n_m})| - n_{A_l+1} + 1\Big\}$ be the move in which this happens.

Proposition 4.2.4. *Let $K_{n_1,\dots,n_m} = (V, E)$ be a complete multipartite graph and $a = (a_1, \dots, a_k)$, $b = (b_1, \dots, b_k)$ where $a_i, b_i \in \mathbb{N}$ for $1 \leq i \leq k$. Further, let $a_1 < m$ and $a_1 + b_1 < 1 + \sum_{i=1}^{a_1} n_i + \sum_{j=a_1+2}^{m} n_j$.*

- *If Alice fixes the coloring, then it holds*

$$\gamma_g(K_{n_1,\dots,n_m}; a, b) = m - h_g + B_{r_g-1}.$$

- *If Bob fixes the coloring, then it holds*

$$\gamma_g(K_{n_1,\dots,n_m}; a, b) = \Big(\sum_{i=1}^{A_{t_g}} n_i\Big) + \Big(\sum_{j=A_{t_g}+2}^{m} n_j\Big) + 1.$$

Proof. Assume as above that the players apply their respective optimal strategies. Once every independent set contains at least one colored vertex the coloring is fixed. Clearly, either Alice or Bob will fix the coloring.

- Assume Alice fixes the coloring. Clearly she makes r_g moves while Bob makes $r_g - 1$ moves. According to her optimal strategy Alice jumps every time in an uncolored independent set and hence she uses $A_{r_g} - (A_{r_g} - (m - h_g))$ colors. Bob uses B_{r_g-1} colors. Hence, $m - h_g + B_{r_g-1}$ colors are required to guarantee Alice's victory.

- Assume Bob fixes the coloring. Alice uses A_{t_g} colors while Bob uses $\Big(\sum_{i=1}^{A_{t_g}} n_i\Big) - A_{t_g} + \Big(\sum_{j=A_{t_g}+2}^{m} n_j\Big) + 1$ colors. Hence, $\Big(\sum_{i=1}^{A_{t_g}} n_i\Big) + \Big(\sum_{j=A_{t_g}+2}^{m} n_j\Big) + 1$ colors are needed to guarantee Alice's victory. □

4.3 The General Asymmetric Game Chromatic Number of Forests

The aim of the following is to generalize the asymmetric game on forests, investigated by Kierstead in [17]. Throughout the section we will consider the m-tuples $a = (a_1, ..., a_m)$ and $b = (b_1, ..., b_m)$ such that a_t and b_t stand for the number of vertices that are being marked or colored in the tth turn, respectively, for $t \in \{1, ..., m\}$. Let $a_{\max} := \max\limits_{1 \le i \le m} \{a_i\}$ and $b_{\max} := \max\limits_{1 \le i \le m} \{b_i\}$. It is of our interest to determine the general asymmetric game chromatic number for some relevant distributions of the m-tuples. For the purpose of determining the upper bound of $\gamma_g(\mathcal{T}; a, b)$ we define the *general asymmetric marking game* and the *general asymmetric marking game number*.

The General Asymmetric Marking Game on Graphs

We refer to a finite graph $G = (V, E)$ with $|V| = n$ and follow the notations on page 26 regarding the linear ordering L on $V(G)$, the orientation G_L and the in- and outneighborhood.

Definition 4.3.1. Let $a = (a_1, a_2, ..., a_m)$ and $b = (b_1, b_2, ..., b_m)$ with $a_i, b_i \in \mathbb{N}$ for $i \in \{1, ..., m\}$, where $\sum_{i=1}^{m}(a_i + b_i) \ge n$. Further, let $\Pi(G)$ be the set of all linear orderings on $V(G)$. Two players Alice and Bob take turns marking vertices of G from the shrinking set U of unmarked vertices until $U = \emptyset$ with Alice playing first. The tth move of Alice and Bob consists of marking a_t and b_t vertices, respectively. This results in a linear ordering $L \in \Pi(G)$ of the vertices of G with $x < y$ iff x is marked before y. The *ga-score* of the game is equal to $\Delta^+_{G_L} + 1$. Alice wins, if the ga-score is at most a given integer s; otherwise, Bob wins.

For $a = (a_1, a_2, ..., a_m)$ and $b = (b_1, b_2, ..., b_m)$ the *general asymmetric marking game number* $col_g^{ga}(G; a, b)$ of a graph $G = (V, E)$ is the least integer s such that Alice has a winning strategy for the general asymmetric marking game, that is $\Delta^+_{G_L} + 1 \le s$.

Since Alice can apply the strategy of the marking game, while playing the coloring game by coloring with First-Fit, the general asymmetric marking game number is an upper bound of the general asymmetric game chromatic number. Thus, for a graph $G = (V, E)$ it holds

$$\gamma_g(G; a, b) \leq col_g^{ga}(G; a, b).$$

Upper Bounds

For determining the general asymmetric marking game number we apply techniques of Kierstead, worked out in [17], where he proved for the asymmetric marking game number $col_g(\mathcal{F}; a, b) \leq b + 3$ for $b \leq a$. The basic idea of the proof is to simulate the $(1, 1)$-marking game on forests, investigated by Kierstead and Tuza in [18], which shows that $col_g(T; 1, 1) \leq 4$ for any forest T. For the sake of completeness and for better understanding we include the proofs in the appendix (A.3.1).

We will make use of the following notations: Let M^t denote the set of marked vertices and $U^t = V - M^t$ be the set of unmarked vertices after t plays, for $1 \leq t \leq |V|$, where a play consists of marking one vertex. A component S^t of U^t is called an *unmarked component* and $N(S^t)$ denotes the neighborhood of S^t. Obviously $N(S^t) \subseteq M^t$. The *weight* $w(S^t)$ of S^t is defined by $w(S^t) := |N(S^t)|$. Since S^t is a tree, clearly if one of the players marks a vertex $x \in S^t$ so that $N(x) \cap M^t \neq \emptyset$, then every component of $S^t - \{x\}$ has weight at most $w(S^t)$.

Lemma 4.3.2. *Let $T = (V, E)$ be a forest. If $a_{i+1} \geq b_i$ for $i \in \{1, ..., m-1\}$, then $col_g^{ga}(T; a, b) \leq b_{\max} + 3$.*

Proof. We will show that Alice has a winning strategy that results in a ga-score of at most $b_{\max} + 3$ in the general asymmetric marking game on T. Alice's winning strategy is to maintain the following invariant: At the end of each of her turns the weight of every unmarked component is at most 2. Then after Bob's turn no unmarked vertex is adjacent to more than $b_{\max} + 2$ marked vertices and hence by the definition of the ga-score it follows $col_g^{ga}(T; a, b) \leq b_{\max} + 3$ for $T \in \mathcal{F}$. In order to maintain the invariant Alice simulates the $(1, 1)$-marking game, see lemma A.3.1.

Assume it is Alice's $(t+1)$th turn and Bob has just colored b_t vertices. She considers the set M of marked vertices before Bob's last move and the vertices $v_1, ..., v_{b_t}$ Bob has marked during his last turn. Then she proceeds in the following manner:

First she considers the vertex v_1 and asks herself which vertex she would have marked in the $(1,1)$-marking game. If her response would be u_1, she checks if $u_1 \in \{v_1, ..., v_{b_t}\}$.

- If $u_1 \in \{v_1, ..., v_{b_t}\}$, Bob answers his own threat and Alice updates M with $M \cup \{v_1, u_1\}$. Afterwards, she goes on with v_2.
- If $u_1 \notin \{v_1, ..., v_{b_t}\}$ she marks u_1, updates M with $M \cup \{v_1, u_1\}$ and goes on with v_2.

In this manner she goes on with every vertex in $\{v_1, ..., v_{b_t}\}$ which is possible by the assumption $a_{i+1} \geq b_i$ for all $i \in \{1, ..., m-1\}$. Afterwards, she marks vertices considering to maintain the invariant, if necessary. $\square$

We have been working under the assumption that $a_{i+1} \geq b_i$ for all $i \in \{1, ..., m-1\}$. In the remainder of this section we additionally suppose that there exists one j with $a_{j+1} < b_j$ and $a_{i+1} \geq b_i$ for all $i \in \{1, ..., m-1\} \setminus \{j\}$. First, we will consider the restriction $a_i \geq b_i$ for all $i \in \{1, ..., m\}$ and afterwards we will drop it.

For $m, k \in \mathbb{N}$ let $x = (x_1, ..., x_k, ..., x_m)$ be an m-tuple. We denote by $x_{k_{\max}}$ the greatest component of the k-tuple $(x_1, ..., x_k)$.

Lemma 4.3.3. *Let $T = (V, E)$ be a forest where $a_i \geq b_i$ for all $i \in \{1, ..., m\}$. If there exists one j with $a_{j+1} < b_j$ for $j \in \{1, ..., m-1\}$ and $a_{i+1} \geq b_i$ for all $i \in \{1, ..., m-1\} \setminus \{j\}$, then*

$$col_g^{ga}(T; a, b) \leq b_{\max} + 3.$$

Proof. Without loss of generality assume that $b_l = b_{max}$. Let $l \leq j$ such that $b = (b_1, ..., b_l = b_{max}, ..., b_j, ..., b_m)$. We prove a winning strategy for Alice which results in a ga-score of at most $b_{max} + 3$ in the general asymmetric marking game on T.

Alice's strategy is to maintain the same invariant as in lemma 4.3.2 and again to simulate the $(1,1)$-marking game. Hence, her intention is to achieve that at the end of each of her turns the weight of every unmarked component is at most 2. Then after Bob's jth turn, no unmarked vertex is adjacent to more than $2 + b_{j_{max}}$ marked vertices. Assume that Bob has played his jth turn and colored vertices $v_1, ..., v_{b_j}$ such that there is a component S' which has weight $2 + b_j$. Since it holds $a_{j+1} < b_j$, Alice cannot achieve her invariant after her $(j+1)$th turn. Hence, the weight of every unmarked component is at most $2 + b_j - a_{j+1}$. The following calculations demonstrate the maximum weight of the unmarked components of T after Bob has played his $(k+1)$th and $(k+2)$th turn, respectively, where x denotes the maximum weight after the $(k+1)$th turn and y denotes the maximum weight after the $(k+2)$th turn:

$$x = \underbrace{2 + (b_j - a_{j+1}) + (b_{j+1} - a_{j+2}) + ... + (b_k - a_{k+1}) + b_{k+1}}_{\text{after the } (k+1)\text{th turn}},$$

$$y = \underbrace{2 + (b_j - a_{j+1}) + (b_{j+1} - a_{j+2}) + ... + (b_k - a_{k+1}) + (b_{k+1} - a_{k+2}) + b_{k+2}}_{\text{after the } (k+2)\text{th turn}}.$$

By the assumption $a_i \geq b_i$ for all $i \in \{1, ..., m\}$ it holds:

$$\begin{aligned} y &= 2 + (b_j - a_{j+1}) + (b_{j+1} - a_{j+2}) + ... + (b_k - a_{k+1}) + (b_{k+1} - a_{k+2}) + b_{k+2} \\ &= 2 + (b_j - a_{j+1}) + (b_{j+1} - a_{j+2}) + ... + (b_k - a_{k+1}) + b_{k+1} + \underbrace{(b_{k+2} - a_{k+2})}_{\leq 0} \\ &\leq 2 + (b_j - a_{j+1}) + (b_{j+1} - a_{j+2}) + ... + (b_k - a_{k+1}) + b_{k+1} = x. \end{aligned}$$

Thus, we can conclude that the weight of every unmarked component is decreasing after Bob's jth turn.

Let $l > j$ such that $b = (b_1, ..., b_j, ..., b_l = b_{max}, ..., b_m)$. By the assumption $a_{i+1} \geq b_i$ for all $i \in \{j+1, ..., m-1\}$ after Alice's $j+1$th turn every unmarked component has weight at most $2 + b_j - a_{j+1}$. Further since $a_i \geq b_i$ for all $i \in \{1, ..., m\}$ it holds $a_l \geq b_l > b_{l-1}$. Thus after her $l$the turn every unmarked component has weight at most 2. □

Now it is of our interest to restrict our attention to the case that the assumption $a_i \geq b_i$ for all $i \in \{1, ..., m\}$ is not given. Then obviously Alice's method of decreasing the weight of every unmarked component breaks down, since $a_i - b_i \geq 0$ does not hold for every $i \in \{1, ..., m\}$. Thus, we worked out the following:

Corollary 4.3.4. *Let $T = (V, E)$ be a forest. If there exists one j with $a_{j+1} < b_j$ for $j \in \{1, ..., m-1\}$ and $a_{i+1} \geq b_i$ and for all $i \in \{1, ..., m-1\} \setminus \{j\}$, then*

$$col_g^{ga}(T; a, b) \leq \max\{\max_{j \leq l-1 \leq m} \sum_{i=j}^{l-1} (b_i - a_{i+1}) + b_l + 3\ , b_{\max} + 3\}.$$

Proof. If $b_{\max} + 3 \geq \sum_{i=j}^{l-1} (b_i - a_{i+1}) + b_l + 3$, then we can refer to the lemma 4.3.2. Thus, we assume that $\sum_{i=j}^{l-1} (b_i - a_{i+1}) + b_l + 3 > b_{\max} + 3$. Our purpose is to prove a winning strategy for Alice which results in a ga-score of at most $\sum_{i=j}^{l-1} (b_i - a_{i+1}) + b_l + 3$ in the general asymmetric marking game.

Without loss of generality assume that $a_2 \geq b_1, a_3 \geq b_2, ..., a_j \geq b_{j-1}, a_{j+1} < b_j, a_{j+2} \geq b_{j+1}, ..., a_n \geq b_{n-1}$. Then again as in the lemma 4.3.3 Alice's winning strategy is to simulate the $(1, 1)$-marking game in order to achieve that at the end of each of her turns the weight of every unmarked component is at most 2 until she has played her $(j+1)$th turn, where the weight of every unmarked component is at most $2 + b_j - a_{j+1}$. However, in contrast to the case that $a_i \geq b_i$ for all $i \in \{1, ...m\}$, the weight of every unmarked component does not decrease necessarily, after Alice's $(j+1)$th turn. Hence, the ga-score is bounded by $\sum_{i=j}^{l-1} (b_i - a_{i+1}) + b_l + 3$. □

Lower Bounds

In [17] Kierstead worked out a strategy for Bob such that the asymmetric game chromatic number of a forest T is at least $b + 3$, if $a < 2b$, where Alice colors a and Bob b vertices in a row. In particular, he determined that the exactly number of turns Bob needs in order to achieve this result is $2b + 3$. Moreover, he proved that the asymmetric marking game number of a forest is $b+3$ for the case $a < 3b$ and gave a winning strategy for Bob after $9b^2 + 3b + 2$ turns. For the sake of completeness we review these strategies, denoted by σ and σ' in A.3.2. We will extend this results, while considering the general asymmetric game on forests for some relevant distributions of the m-tuples $a = (a_1, ..., a_m)$ and

$b = (b_1, ..., b_m)$. It is to be expected that the lower bounds as well as the number of the turns Bob needs in order to win the game differ from Kierstead's results.

Kierstead considered the following graph: Let $T = T_n$ be a forest with vertex set $V(T_n) = \bigcup_{i=1}^{20} [n]^i$ where $[n]$ denotes the set $\{1, ..., n\}$. Further, the edges are pairs of the form $e = (x_1, ..., x_i)(x_1, ..., x_{i+1})$. For $i \in [n]$ the vertex (i) is called a *root* of T_n. For $x \in V(T_n)$ the *weight* of x is defined by the number of distinct colors that are assigned to the neighbors of x. An uncolored vertex is called *dangerous*, if its weight is at least 2.

In the following we will assume that $n \geq l(a_{\max} + b_{\max})$ holds, where l is the number of turns Bob needs in order to win the game. We recall the following definition: The *distance* $d_{(x,y)}$ between two vertices x and y is the number of the edges on the shortest path between x and y.

Lemma 4.3.5. *Let $T = (V, E)$ be a forest. If $a_1 < 2 \cdot b_1$, where $a_1 = ... = a_m$ and $b_i = 2^{i-1} \cdot b_1$ for $i \in \{1, ..., m\}$, then*

$$\gamma_g(T; a, b) \geq b_{\lceil \log_2(2b_1+2) \rceil + 1} + 3.$$

Proof. Assume that $b_{\lceil \log_2(2b_1+2) \rceil + 1} + 2$ colors are given. We will give a winning strategy for Bob. Assume Alice colored a_1 arbitrary vertices and it is Bob's first turn. Then he colors some root x with the color α. In Kierstead's algorithm σ, see A.3.2, Bob colored in his next $2b$ turns $2b^2$ vertices $\{v_1, ..., v_{2b^2}\}$ with α, such that $d_{(x,v_i)} = 3$ and $d_{(v_i,v_j)} = 6$ for all distinct $i, j \in [2b^2]$. Our goal is to determine the number of turns Bob needs in order to color at least the same $2b_1^2$ vertices with the given m-tuple:

Let j be the number of turns after Bob has colored $2b_1^2$ vertices from his second turn. Since the elements of the m-tuple $b = (b_1, ..., b_m)$ are strictly increasing, it holds that $j \leq 2b_1$. Hence, we have

$$2b_1^2 \leq 2^1 \cdot b_1 + 2^2 \cdot b_1 + ... + 2^{j-1} \cdot b_1 \Leftrightarrow 2b_1 \leq \sum_{k=1}^{j-1} 2^k.$$

Since $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$, it follows that

$$2b_1 \leq 2^j - 2 \Leftrightarrow 2b_1 + 2 \leq 2^j \Leftrightarrow \log_2(2b_1 + 2) \leq j.$$

Since $j \in \mathbb{N}$ we can draw the conclusion that after the $1 + \lceil \log_2(2b_1 + 2) \rceil - 1$=$\lceil \log_2(2b_1+2) \rceil$ th turn Bob has colored a root x and among others $2b_1^2$ vertices $v_1, ..., v_{2b_1^2}$ with α, such that $d_{(x,v_i)} = 3$ and $d_{(v_i,v_j)} = 6$ hold. Let $\mathcal{P}$ be the set of the paths $\{x...v_i\}$, for $i \in \{1, ..., 2b_1^2\}$. In particular, the remaining vertices on such a path are uncolored.

By the assumption $a_1 < 2b_1$ and since $j \leq 2b_1$, at the time when Bob has colored $2b_1^2$ vertices, Alice has colored at most

$$(\lceil \log_2(2b_1 + 2) \rceil - 1) \cdot a_1 \leq 2b_1 a_1 \leq 2b_1(2b_1 - 1) = 4b_1^2 - 2b_1$$

of the $4b_1^2$ internal vertices of paths in $\mathcal{P}$. Thus, after Alice's turn there is a subset $\mathcal{P}' \subset \mathcal{P}$ with $|\mathcal{P}'| \geq b_1$, such that each path in $\mathcal{P}'$ contains an uncolored vertex.

- Assume that there exists a path p from $\mathcal{P}'$ with y being the only uncolored vertex. Since the root x and the endpoint of p are colored with α and $d_{(x,v_i)} = 3$ for all $i \in \{1, ..., 2b_1^2\}$, y is dangerous. Thus, Bob wins the game, since in his next turn he will color $b_{j+1} = b_{\lceil \log_2(2b_1+2) \rceil + 1}$ neighbors of y with new distinct colors. This implies that the weight of y is $b_{j+1} + 2 = b_{\lceil \log_2(2b_1+2) \rceil + 1} + 2$, such that at least $b_{j+1} + 3 = b_{\lceil \log_2(2b_1+2) \rceil + 1} + 3$ colors are necessary for achieving a proper coloring of T.

- Otherwise, assume that both vertices of the paths in $\mathcal{P}'$ are uncolored. In his next turn Bob colors $b_{j+1} = b_{\lceil \log_2(2b_1+2) \rceil + 1}$ vertices. In particular, he colors on each path from $\mathcal{P}'$ an external neighbor of an internal vertex with $\beta \neq \alpha$, which creates at least b_1 dangerous vertices. During her next turn Alice has to color each of the new dangerous vertices, which creates another b_1 dangerous vertices that she cannot color because of the assumption $a_1 < 2b_1$. Hence, in his next turn, Bob colors $b_{j+2} = b_{\lceil \log_2(2b_1+2) \rceil + 2}$ external neighbors of a dangerous vertex with distinct colors, such that the weight of such a dangerous vertex is $b_{j+2} + 2 = b_{\lceil \log_2(2b_1+2) \rceil + 2} + 2$. Thus, Bob wins the game after $j + 2 = \lceil \log_2(2b_1 + 2) \rceil + 2$ turns. □

Remark 4.3.6. If we drop the assumption $a_1 = ... = a_m$, then lemma 4.3.5 still holds, if we assume that $a_i < 2b_i$ and $a_{j+1} < 2b_j$ for all $i \in \{1, ..., m\}$,

$j \in \{1, ..., m-1\}$. Thus, the case that each path from $\mathcal{P}'$ with two uncolored vertices still works: Suppose in his last turn Bob created b_j dangerous vertices by coloring b_j external neighbors of uncolored internal vertices of $\mathcal{P}'$ with a distinct color than α. Obviously Alice creates another b_j dangerous vertices, if she colors these b_j vertices made dangerous by Bob. Since $a_{j+1} < 2b_j$ for all $j \in \{1, ..., m-1\}$, Alice cannot color all new dangerous vertices and looses the game with the given set of colors.

We proceed to determine the general asymmetric game chromatic number for further distributions of the m-tuple $a = (a_1, ..., a_m)$ and $b = (b_1, ..., b_m)$with $a_1 = ... = a_m$. In particular, we will consider the assumption $a_1 < 2b_1$ and b strictly increasing. Since the proofs run by the same method as in the proof of lemma 4.3.5, we will briefly sketch it. However, we are interested in calculating the number of turns Bob needs in order to color some root x and at least $2b_1^2$ vertices $v_1, ..., v_{2b_1^2}$ with $d_{(x,v_i)} = 3$ and $d_{(v_i,v_k)} = 6$ for $i, k \in \{1, ..., 2b_1^2\}$, denoted by j_0. Since the basic idea of the proof is the same as in the proof of lemma 4.3.5, we can conclude that the general asymmetric game chromatic number of T is at least $b_{j_0+1} + 3$.

Corollary 4.3.7. *Let $T = (V, E)$ be a forest and $a_1 < 2 \cdot b_1$ with $a_1 = ... = a_m$.*

(i) If $b_i = q^{i-1} \cdot b_1$ for $q \in \mathbb{N}$, $q \neq 1$ and $i \in \{1, ..., m\}$, then

$$\gamma_g(T; a, b) \geq b_{\left\lceil \log_q \left(1-(2b_1+1)(1-q)\right)\right\rceil+1} + 3.$$

(ii) If $b_i = (2i-1) \cdot b_1$ for $i \in \{1, ..., m\}$, then

$$\gamma_g(T; a, b) \geq b_{\left\lceil \sqrt{2b_1+1}\right\rceil+1} + 3.$$

Proof. (i) Let $b_i = q^{i-1} \cdot b_1$ for $q \in \mathbb{N}$, $q \neq 1$ and $i \in \{1, ..., m\}$. Then it holds

$$2b_1^2 \leq qb_1 + q^2 b_1 + ... + q^{j_0-1} b_1 \Leftrightarrow 2b_1 \leq q + q^2 + ... + q^{j_0-1}.$$

For the geometrical series it holds $1 + q + q^2 + ... + q^n = \frac{1-q^{n+1}}{1-q}$. Then

$$2b_1 \leq \frac{1-q^{j_0}}{1-q} - 1 \Leftrightarrow 2b_1 + 1 \leq \frac{1-q^{j_0}}{1-q}.$$

Since $q \in \mathbb{N}$ with $q \neq 1$, it holds

$$(2b_1+1)(1-q) \geq 1-q^{j_0} \Leftrightarrow q^{j_0} \geq 1-(2b_1+1)(1-q) \Leftrightarrow j_0 \geq \log_q \big(1-(2b_1+1)(1-q)\big).$$

Since $j_0 \in \mathbb{N}$, $j_0 = \big\lceil \log_q \big(1-(2b_1+1)(1-q)\big)\big\rceil$. Thus, we can conclude that

$$\gamma_g(T;a,b) \geq b_{\left\lceil \log_q \left(1-(2b_1+1)(1-q)\right)\right\rceil+1} + 3.$$

(ii) Let $b_i = (2i-1)\cdot b_1$ for $i \in \{1,...,m\}$. Then it holds

$$2b_1^2 \leq 3b_1 + ... + (2j_0-1)b_1 \Leftrightarrow 2b_1 \leq 3 + ... + (2j_0-1).$$

From $1+3+5+...+(2n-1) = n^2$ for $n \in \mathbb{N}$, we have

$$2b_1 \leq j_0^2 - 1 \Leftrightarrow \sqrt{2b_1+1} \leq j_0.$$

Since $j_0 \in \mathbb{N}$, $j_0 = \lceil \sqrt{2b_1+1} \rceil$. Thus, we can conclude that

$$\gamma_g(T;a,b) \geq b_{\lceil \sqrt{2b_1+1}\rceil+1} + 3.$$

□

Now we will restrict our attention to some distributions of $b = (b_1,...,b_m)$ under the assumption $a_1 < 3b_1$, where the elements of b are strictly increasing and $a_1 = ... = a_m$. In particular, we will prove the lower bound of the general asymmetric marking game number of forests. Since in each case Bob applies strategy σ', see A.3.2, we will sketch briefly the proof for the general case. Thus, we focus on the number of turns Bob needs in order to win the game. For this purpose, we refer to the notations used for strategy σ' and introduce some new notations:

Denote by j_0 the number of the turns after Bob has marked some root x and the vertices $\{v_1,...,v_{9b^3}\}$, where $d_{(x,v_i)} = 4$ and $d_{(v_i,v_j)} = 8$, for $i,j \in \{1,...,9b^3\}$. Since it holds that $j_0 \leq 9b_1^2$ and because of the assumption $a_1 < 3b_1$, Alice has marked at most

$$(j_0-1)a_1 \leq 9b_1^2 \cdot a_1 \leq 9b_1^2(3b_1-1) = 27b_1^3 - 9b_1^2$$

of the $27b_1^3$ internal vertices, simultaneously. Thus, $|\mathcal{P}'| \geq 3b_1^2$.

- If $p' \in \mathcal{P}'$ does not contain two consecutive unmarked vertices, then after $j_0 + 1$ turns the ga-score is $b_{j_0+1} + 3$.

- Otherwise, assume that $p' \in \mathcal{P}'$ contains at least two consecutive unmarked vertices. Let j_0' be the number of the turns Bob needs in order to mark the next $3b_1^2$ vertices and let $k_0 = j_0 + j_0'$. Since $j_0' \leq 3b_1$ and because of the assumption $a_1 < 3b_1$, we can conclude that Alice marks

$$j_0' a_1 \leq 3b_1 a_1 \leq 3b_1(3b_1 - 1) = 9b_1^2 - 3b_1$$

vertices. This implies that $|\tilde{\mathcal{P}}'| \geq b_1$. Moreover, in the $j_0 + j_0' + 2 = k_0 + 2$th turn the ga-score is $b_{k_0+2} + 3$. Since $b_{j_0+1} + 3 < b_{k_0+2} + 3$ we can draw the conclusion that Bob achieves a ga-score of $b_{j_0+1} + 3$.

The procedure of the following is to determine j_0 and k_0 for the given distributions of b.

Corollary 4.3.8. *Let $T = (V, E)$ be a forest and $a_1 < 3 \cdot b_1$ with $a_1 = ... = a_m$.*

(i) If $b_i = 2^{i-1} \cdot b_1$ for $i \in \{1, ..., m\}$, then

$$col_g(T; a, b) \geq b_{\lceil \log_2(9b_1^2+2) \rceil + 1} + 3.$$

(ii) If $b_i = q^{i-1} \cdot b_1$ for $q \in \mathbb{N}$, $q \neq 1$ and $i \in \{1, ..., m\}$, then

$$col_g(T; a, b) \geq b_{\lceil \log_q(1-(9b_1^2+1)(1-q)) \rceil + 1} + 3.$$

(iii) If $b_i = (2i - 1) \cdot b_1$ for $i \in \{1, ..., m\}$, then

$$col_g(T; a, b) \geq b_{\left\lceil \sqrt{9b_1^2+1} \right\rceil + 1} + 3.$$

Proof. (i) Let $b_i = 2^{i-1} \cdot b_1$ for $i \in \{1, ..., m\}$.

Determine j_0:

$$9b_1^3 \leq 2^1 \cdot b_1 + 2^2 \cdot b_1 + ... + 2^{j_0-1} \cdot b_1 \Leftrightarrow 9b_1^2 \leq \sum_{i=1}^{j_0-1} 2^i.$$

Since $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$, it follows that

$$9b_1^2 \leq 2^{j_0} - 2 \Leftrightarrow 9b_1^2 + 2 \leq 2^{j_0} \Leftrightarrow \log_2(9b_1^2 + 2) \leq j_0.$$

Since $j_0 \in \mathbb{N}$, we have $j_0 = \lceil \log_2(9b_1^2 + 2) \rceil$.

Determine k_0:

$$3b_1^2 \leq b_{j_0+1} + ... + b_{k_0} \Leftrightarrow 3b_1^2 \leq 2^{j_0} \cdot b_1 + ... + 2^{k_0-1} \cdot b_1 \Leftrightarrow$$

$$3b_1 \leq 2^{j_0} + ... + 2^{k_0-1} \Leftrightarrow 3b_1 \leq \sum_{i=0}^{k_0-1} 2^i - \sum_{l=0}^{j_0-1} 2^l.$$

Since $\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$, it follows that

$$3b_1 \leq (2^{k_0} - 1) - (2^{j_0} - 1) = 2^{k_0} - 2^{j_0} \Leftrightarrow 2^{k_0} \geq 3b_1 + 2^{j_0} \Leftrightarrow k_0 \geq \log_2(3b_1 + 2^{j_0}).$$

Since $k_0 \in \mathbb{N}$, we have $k_0 = \lceil \log_2(3b_1 + 2^{j_0}) \rceil$.

Thus, we can draw the conclusion that after $\lceil \log_2(3b_1 + 2^{j_0}) \rceil + 2$ turns the ga-score of the game is $b_{\lceil \log_2(3b_1+2^{j_0}) \rceil+2} + 3$ for $j_0 = \lceil \log_2(9b_1^2 + 2) \rceil$. It follows that

$$col_g^{ga}(T; a, b) \geq b_{\lceil \log_2(9b_1^2+2) \rceil+1} + 3.$$

(ii) Let $b_i = q^{i-1} \cdot b_1$ for $q \in \mathbb{N}$, $q \neq 1$ and $i \in \{1, ..., m\}$.

Determine j_0:

$$9b_1^3 \leq qb_1 + ... + q^{j_0-1}b_1 \Leftrightarrow 9b_1^2 \leq q + ... + q^{j_0-1}.$$

For the geometrical series it holds $1 + q + q^2 + ... + q^n = \frac{1-q^{n+1}}{1-q}$. Then

$$9b_1^2 \leq \frac{1 - q^{j_0}}{1 - q} - 1 \Leftrightarrow 9b_1^2 + 1 \leq \frac{1 - q^{j_0}}{1 - q}.$$

Since $q > 1$, it holds

$$(9b_1^2+1)(1-q) \geq 1-q^{j_0} \Leftrightarrow q^{j_0} \geq 1-(9b_1^2+1)(1-q) \Leftrightarrow j_0 \geq \log_q(1-(9b_1^2+1)(1-q)).$$

Since $j_0 \in \mathbb{N}$, we have $j_0 = \lceil \log_q(1 - (9b_1^2 + 1)(1 - q)) \rceil$.

Determine k_0:

$$3b_1^2 \leq b_{j_0+1} + ... + b_{k_0} \Leftrightarrow 3b_1^2 \leq q^{j_0}b_1 + ... + q^{k_0-1}b_1 \Leftrightarrow 3b_1 \leq \sum_{i=0}^{k_0-1} q^i - \sum_{l=0}^{j_0-1} q^l \Leftrightarrow$$

$$3b_1 \leq \Big(\frac{1-q^{k_0}}{1-q} - 1\Big) - \Big(\frac{1-q^{j_0}}{1-q} - 1\Big) \Leftrightarrow 3b_1 \leq \frac{q^{j_0} - q^{k_0}}{1-q}.$$

Since $q > 1$, we have

$$3b_1(1-q) \geq q^{j_0} - q^{k_0} \Leftrightarrow q^{k_0} \geq q^{j_0} - 3b_1(1-q) \Leftrightarrow k_0 \geq \log_q(q^{j_0} - 3b_1(1-q)).$$

Since $k_0 \in \mathbb{N}$, it holds that $k_0 = \lceil \log_q(q^{j_0} - 3b_1(1-q)) \rceil$.

Thus, we can draw the conclusion that after $\lceil \log_q(q^{j_0} - 3b_1(1-q)) \rceil + 2$ turns the ga-score of the game is $b_{\lceil \log_q(q^{j_0}-3b_1(1-q)) \rceil+2} + 3$ for $j_0 = \lceil \log_q(1-(9b_1^2+1)(1-q)) \rceil$. It follows that

$$col_g^{ga}(T; a, b) \geq b_{\lceil \log_q(1-(9b_1^2+1)(1-q)) \rceil+1} + 3.$$

(iii) Let $b_i = (2i-1) \cdot b_1$ for $i \in \{1, ..., m\}$.

Determine j_0:

$$9b_1^3 \leq 3b_1 + ... + (2j_0 - 1)b_1 \Leftrightarrow 9b_1^2 \leq 3 + ... + (2j_0 - 1).$$

From $1 + 3 + 5 + ... + (2n-1) = n^2$ for $n \in \mathbb{N}$, we have

$$9b_1^2 \leq j_0^2 - 1 \Leftrightarrow \sqrt{9b_1^2 + 1} \leq j_0.$$

Since $j_0 \in \mathbb{N}$, it holds $j_0 = \left\lceil \sqrt{9b_1^2 + 1} \right\rceil$.

Determine k_0:

$$3b_1^2 \leq b_{j_0+1} + ... + b_{k_0} \Leftrightarrow 3b_1^2 \leq (2(j_0+1) - 1)b_1 + ... + (2k_0 - 1)b_1 \Leftrightarrow$$

$$3b_1 \leq (2(j_0+1) - 1) + ... + (2k_0 - 1) \Leftrightarrow 3b_1 \leq k_0^2 - j_0^2 \Leftrightarrow \sqrt{3b_1 + j_0^2} \leq k_0.$$

Since $k_0 \in \mathbb{N}$, it holds that $k_0 = \left\lceil \sqrt{3b_1 + j^2} \right\rceil$.

Thus, we can draw the conclusion that after $\left\lceil \sqrt{3b_1 + j_0^2} \right\rceil + 2$ turns the ga-score of the game is $b_{\left\lceil \sqrt{3b_1+j_0^2} \right\rceil +2} + 3$ for $j_0 = \left\lceil \sqrt{9b_1^2 + 1} \right\rceil$. It follows that

$$col_g^{ga}(T; a, b) \geq b_{\left\lceil \sqrt{9b_1^2+1} \right\rceil +1} + 3.$$

$\square$

Appendix A

Appendix

A.1 (k, d)-Coloring and r-Interval Coloring of Graphs

In [20] Vince introduced the (k, d)-coloring of a graph G as follows:

Definition A.1.1. Given two positive integers k, d such that $k \geq d$. A (k, d)-*coloring* of a graph $G = (V, E)$ is an assignment ϕ of colors $\{0, 1, ..., k-1\}$ to the vertices of G such that for any two adjacent vertices v_1 and v_2 we have $d \leq |\phi(v_1) - \phi(v_2)| \leq k - d$. The *star-chromatic number* $\chi^*(G)$ of a graph G is the infimum of the ratio $\frac{k}{d}$ for which there exists a (k, d)-coloring of G.

An equivalent definition called the r-interval coloring is due to Zhu, pointed out in [22]:

Definition A.1.2. Let r be a rational number, and let $I = [0, r)$ be an interval of length r. For a graph $G = (V, E)$, an *r-interval coloring* g of G is a mapping $g : V \to [0, r)$ such that for every edge (x, y) of G, $1 \leq |g(x) - g(y)| \leq r - 1$ holds. For $0 \leq x < r$, let $|x|_r = \min\{|x|, r - |x|\}$. Then an r-interval coloring g of G is a mapping $g : V(G) \to [0, r)$ such that $|g(x) - g(y)|_r \geq 1$ for every edge (x, y) of G. The *interval-chromatic number* $\chi_I(G)$ is the infimum of those r for which G has an r-interval coloring.

A.2 On the Game Chromatic Number of Trees

Theorem A.2.1. *Let $T = (V, E)$ be a tree, then $\gamma(T) \leq 4$.*

Proof. Assume 4 colors are given. Initially, Alice colors an arbitrary vertex $r \in V(T)$. Hence, r is called the *root*. During the whole game Alice maintains a subtree T_0 of T that contains all the vertices colored so far. Alice initializes $T_0 = \{r\}$.

Suppose Bob has just colored vertex v. Let P be the directed path from r to v in T and let u be the last vertex P has in common with T_0. Then Alice does the following:

(1) Update $T_0 := T \cup P$.

(2) If u is uncolored, assign a feasible color to u.

(3) If u is colored and T_0 contains an uncolored vertex $v \in T_0$, assign a feasible color to v.

(4) If all vertices in T_0 are colored, color any vertex v adjacent to T_0 and update $T_0 := T_0 \cup \{v\}$.

This strategy of Alice guarantees each player the existence of an uncolored vertex with at most 3 colored neighbors until the whole tree is colored. □

A.3 The Asymmetric Game for the Class of Forests

A.3.1 Upper Bounds

Lemma A.3.1. *Let $T = (V, E)$ be a forest. Then $col_g(T; 1, 1) \leq 4$.*

Proof. Alice's strategy is to maintain the following invariant: At the end of each of Alice's moves, the weight of every unmarked component is at most 2. Obviously, if Alice achieves this, after Bob's move no unmarked vertex is adjacent to more than 3 marked vertices.

Further, Alice is able to maintain the invariant.

- After Alice's first move the invariant clearly holds.

- Suppose the invariant holds after Alice's last move and it is Bob's turn. Assume Bob marks a vertex v in an unmarked component S of $V - M$, where M is the set of marked vertices. The worst case is if Bob's move produces an unmarked component S' of $V - (M - \{v\})$ with weight 3. Then $S' \subset S$ and $N(S) \cup \{v\} = N(S')$. Since T is a forest, 2 marked vertices cannot each be adjacent to two distinct unmarked components. Hence, there exists at most one unmarked component of $V - (M \cup \{v\})$ with weight 3. Obviously because T is a forest, there exists a vertex $u \in S'$ such that if u is marked, every unmarked component of $S' - \{u\}$ has weight at most 2.

  Alice plays as follows: If there exists an unmarked component S^* with weight 3, Alice chooses it and marks a vertex $u \in S^*$ so that every unmarked component of $S^* - \{u\}$ has weight at most 2. Otherwise, she chooses any unmarked component of $V - (M \cup \{v\})$ and marks a vertex so that every unmarked component has weight at most 2. □

Lemma A.3.2. *If $b \leq a$, then $col_g(\mathcal{F}; a, b) \leq b + 3$.*

Proof. Consider the (a, b)-marking game. We show that Alice can still maintain the invariant. Then after any of her turns, no unmarked vertex is adjacent to more than two marked vertices. It follows that after any of Bob's turns no unmarked vertex is adjacent to more than $b + 2$ marked vertices. To maintain the invariant, Alice simulates the $(1, 1)$-marking game. She considers the set M of marked vertices after her last turn together with the last b vertices $v_1, ..., v_b$ that Bob has marked. First she sets $i := 1$. Then she asks herself where she would have played in the $(1, 1)$-marking game if Bob has just marked v_i. Then she updates $M := M \cup \{v_i\}$. If her response would be u_i then she checks to see whether $u_i \in \{v_{i+1}, ..., v_b\}$. If so, by reordering if necessary, assume $u_i = v_{i+1}$. In this case, Bob has answered his own threat, and she updates $M := M \cup \{v_{i+1}\}$ and goes on with $i + 2$. Otherwise, she marks u_i and updates $M := M \cup \{u_i\}$ and goes on with $i+1$. Continuing in this fashion until $i > b$, she marks vertices when necessary to respond the threats by Bob that he does not answer himself.

Eventually, when $i > b$, she will have marked at most $b \leq a$ vertices. If she has not yet marked a vertices she continues as though Bob was playing by passing in the $(1,1)$- marking game. In this way, she maintains the invariant. □

A.3.2 Lower Bounds

Strategy σ
Bob's strategy is to force Alice to leave an uncolored vertex v dangerous, such that he can color the remaining uncolored neighbors of v with b new distinct colors which obviously results in a asymmetric game chromatic number at least $b+3$. Assume $b+2$ colors are given.

- 1*st turn*: Among others Bob colors some root x with the color α.
- *next* $2b$ *turns*: Throughout the following $2b$ turns he colors vertices $\{v_1, ..., v_{2b^2}\}$ with the color α, where $d_{(x,v_i)} = 3$, $d_{(v_i,v_j)} = 6$ for all $i, j \in [2b^2]$. In particular, x is the only colored vertex on the $x...v_i$ path in T before Bob colors v_i. Simultaneously Alice has colored at most $2ba$ of the $4b^2$ internal vertices of the paths from $\mathcal{P}$, where $\mathcal{P}$ is the set of the $x...v_i$ paths for $i \in [2b^2]$.
- $(2b+2)$*th turn*: Because of the assumption $a < 2b$ there is a subset $\mathcal{P}' \subset \mathcal{P}$ with $|\mathcal{P}'| = b$ so that the paths in $\mathcal{P}'$ contain at least one uncolored vertex, respectively. Let y be the only uncolored vertex on such a path. Then Bob wins the game since y is dangerous. Otherwise, assume that both internal vertices of each path in $\mathcal{P}'$ are uncolored. Then Bob colors an external neighbor of one internal vertex of each path from $\mathcal{P}'$ with color $\beta \neq \alpha$, respectively. This creates b dangerous vertices. However, during her next turn Alice must color each of the b dangerous vertices, which creates another b dangerous vertices that she cannot color since $a < 2b$. Without loss of generality assume she used the color γ.
- $(2b+3)$*th turn*: Let u be a dangerous vertex. Bob colors b neighbored vertices of u with distinct colors unequal to α and γ, which implies that there is no feasible color left for u.

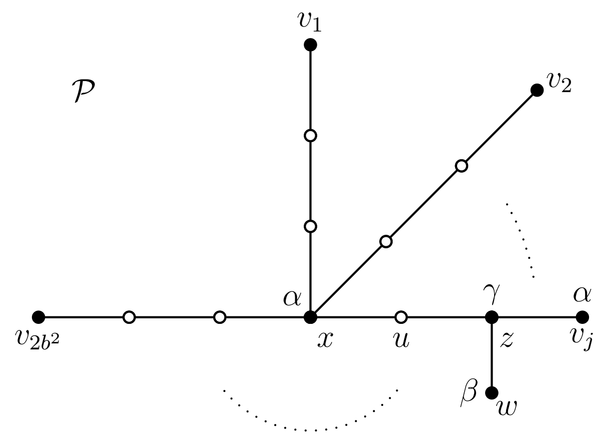


FIGURE: The coloring of the path $x \to v_j$ from $\mathcal{P}' \subseteq \mathcal{P}$, where Bob colored w with β in his $(2b+2)$th turn and Alice colored vertex z with γ, such that u is dangerous now!

Strategy σ'

As in strategy σ, it is Bob's purpose to force Alice to leave a dangerous vertex v at the end of a turn. Then he can mark b additional unmarked neighbors of v and obtain by the definition of the score that $col_g(T; a, b) \geq b+3$ after $9b^2+3b+2$ turns. Assume Alice has marked a vertices in her first turn and it is Bob's turn.

- 1*st turn:* Among others Bob marks some root x.

- *next* $9b^2$ *turns:* Bob marks $9b^3$ vertices $\{v_1, ..., v_{9b^3}\}$ such that $d_{(x,v_i)} = 4$ and $d_{(v_i,v_j)} = 8$ for all $i, j \in \{1, ..., 9b^3\}$. This creates a set $\mathcal{P}$ of $9b^3$ paths. In particular, the $27b^3$ internal vertices of each path from $\mathcal{P}$ are unmarked. Simultaneously Alice marks at most $9b^2a \leq 9b^2(3b-1) = 27b^3 - 9b^2$ of the $27b^3$ internal vertices on the paths in $\mathcal{P}$.

- $(9b^2+2)$*th turn:* Because of the assumption $a < 3b$, there is a $3b^2$-subset $\mathcal{P}' \subset \mathcal{P}$ so that the paths from $\mathcal{P}'$ contain at least one unmarked vertex, respectively. If $p' \in \mathcal{P}'$ does not contain two consecutive unmarked vertices, then p' contains a dangerous vertex and Bob wins. Otherwise, assume that each path p' has at least two consecutive unmarked vertices.

- *next* $3b$ *turns:* If p' has three unmarked vertices, Bob marks an external neighbor of the internal central vertex. If p' has only two consecutive vertices, then Bob marks the external neighbor of an unmarked external neighbor of the internal central vertex. In each case he creates a set $\tilde{\mathcal{P}}$ of paths, such that each path $\tilde{p}$ has three unmarked vertices, where each of them has an unmarked neighbor.

Simultaneously Alice marks $3ba \leq 3b(3b-1) = 9b^2 - 3b$ vertices. If she marks any vertex on $\tilde{p}$, then she has to mark all three unmarked vertices of $\tilde{p}$ in a row. Otherwise, she would leave a dangerous vertex.

- $(9b^2+3b+1)$*th turn:* Since $a < 3b$, there is a b-subset $\tilde{\mathcal{P}}' \subset \tilde{\mathcal{P}}$, such that each path from $\tilde{\mathcal{P}}'$ consists of three unmarked vertices. Bob replies by marking an external neighbor of the central vertex on a path $\tilde{p}' \in \tilde{\mathcal{P}}'$ which creates b dangerous vertices. Then Alice is forced to mark each of the b dangerous vertices. But then she would create $2b$ additional dangerous vertices, which she cannot mark since $a < 3b$. Again Bob wins.

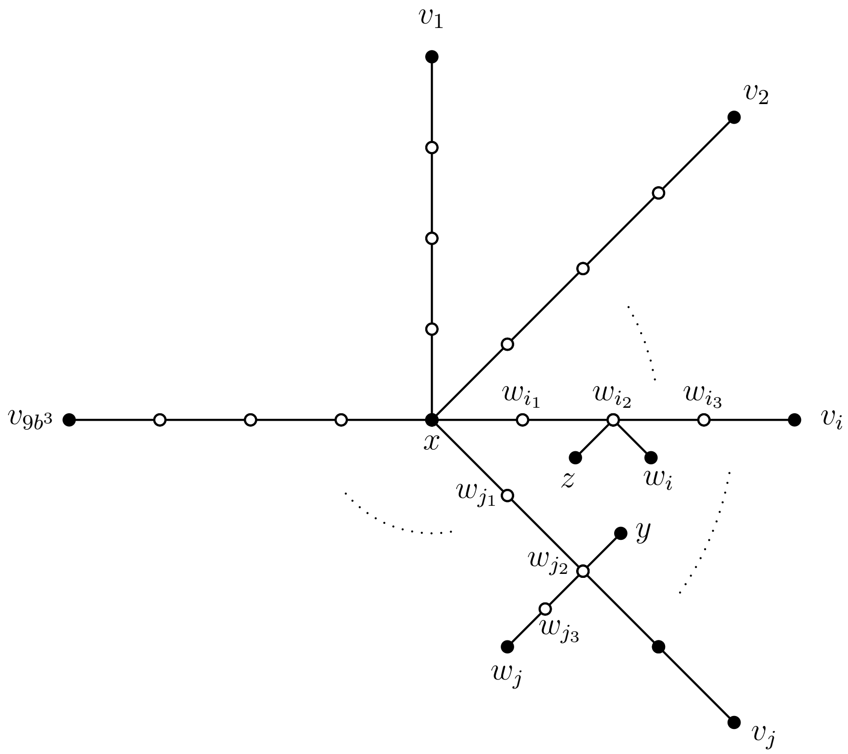


FIGURE: Let $\tilde{p_i'} = w_{i_1}w_{i_2}w_{i_3}$ and $\tilde{p_j'} = w_{j_1}w_{j_2}w_{j_3}$ for $\tilde{p_i'}, \tilde{p_j'} \in \tilde{\mathcal{P}}'$. Assume Bob marks in his $(9b^2+3b+1)$th turn vertices y and z. Then w_{i_2} and w_{j_2} are dangerous now.

Thus, in his next turn Bob wins the game by marking b additional neighbors of the central vertex.

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